On a Class of Inverted Distributions

by

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خلاصة

تظهر مسألة تقدير التوزيعات المعكوسة في عدة حالات منها على سبيل المثال في علم الاقتصاد وعلم الأحياء ومسح العينات وفي العلوم الهندسية وتلاحظ في اختبار العمر الباقي • حاول بعض المؤلفين حديثا لدراسة العزوم لفئة المعكوسة أو العزوم السالبة لتوزيعات معينة •

أقترح شاو وسترودمان طريقة لحساب العزوم السالبة للمتغيرات العشوائية الموجبه عملت محاولة لأحداث فئة جديدة للتوزيعات المعكوسه باستمال اقتراب موحد للتحويلات وعممت الفئة البيرسونيه وبرهنت بعض الميزات الرياضية •

برهن ان دوال الخطر لفئة التوزيعات المعكوسه (المختلفة عن الأصل) تشبه دوال الخطر لتوزيعات اللوغراتميه الطبيعه وبروستين • هذه الظاهرة الفريدة لدوال الخطر توصي بالتوزيعات المعكوسه في العلوم الطبيعه وسيطرة الأمراض ودراسة معدل الوفيات والعمر الباقي •

ABSTRACT

The probability distributions corresponding to the reciprocal transformations arise in different statistical and scientific works, for example in problems related to econometrics, biological sciences, survey sampling, engineering sciences and life testing problems. Some authors have recently attempted to study the moments of the inverted class of certain distributions which can also be interpreted as negative moments of the distributions.

In this thesis an attempt is made to generate a new class of inverted distributions using a unified approach of transformation and to generalize Pearsonian class of distributions. Various mathematical and statistical properties of the inverted distributions are discussed and some mathematical characterizations are obtained.

The hazard functions of the inverted class of distributions, unlike their parent distributions, are shown to have an analogy with the hazard functions of the logarithmic normal and Bernstein distributions. This unique feature of hazard functions suggest the applicability of inverted distributions in medicine sciences i.e; analytical study of disease control, mortality studies and life testing.

1. INTRODUCTION

Due to the present economic and social conditions, the sciences, technology, and other spheres of the society are recognizing ever expanding needs for quantification. The random quantities arising in conceptualization, in modeling, in simulation, in data analysis, in life testing, and in decision making lead to various kinds of problems in the realm of distribution theory and request for solution. Thus statistical distributions remain an important and focal area of study.

The concept of probability models is familiar in applied science work. The unique feature of a probability model is that it describes in mathematical terms the probability aspects of a measurement variable. The probability density function or a cumulative distribution function serves as statistical model of a random phenomenon. In general, a vast majority of well-known distributions, discrete as well as continuous, have been derived analytically from other models independently of their relevance to particular phenomena. The analytical derivation of models often precedes the search for applications. There are several techniques for generating new distributions. The five simple techniques are

(1) transformations

(2) convolution

(3) convergence in distribution

(4) mixing or compounding

(5) derivation from physical situations

These techniques are discussed in standard texts on mathematical statistics [1,2,3]. A statistical distribution corresponding to the reciprocal transformation is called as an inverted distribution. The probability models corresponding to the reciprocal transformations arise in different statistical and scientific works in the fields of econometrics, biological sciences, survey sampling, in engineering sciences and in life testing [4,5,6].

1.1 LITERATURE SURVEY

A little work on inverted distributions seems to have been done by researchers. Some individuals have attempted to study inverted normal distribution [7] which has also been named as Druzhinin model [8,9] or α model [10,11,12,13]. These distributions have been derived under certain engineering situations [12,13,14,15,16,17]. Recently the moments of the inverted distributions also known as negative moments of the respective distributions have also been studied [18,19,20,21,22,23].

In general, if we consider the life of a device subjected to time dependent degradation due to a linear damage process

$$D(t) = D_0 + rt , \qquad t \varepsilon T \tag{1.1}$$

where D_0 = initial level of damage at time $t = 0$ and r is the rate of damage and if the random variable r follows any of the Pearson probability distribution then the distribution of T is the corresponding inverted one. For example, if r is a normal random variate, then T is an inverted normal random variable.

The inverted normal probability density essentially belongs to a bimodal family of distributions. This distribution is a special case of the three parameter Bernstein probability model, which has been successfully used in a variety of situations such as modelling of cutting tools, quality control and replacement theory. The followings are the main results related to Bernstein and Inverted normal model found in literature which have been given in [16].

(1) The three parameter Bernstein probability density function (p.d.f) [7] is

$$g(x) = 1/(2\pi)^{\frac{1}{2}} (\alpha+\beta)(\beta+\alpha x^2)^{-3/2} \exp[-\tfrac{1}{2}(x-c)^2/(\alpha x^2 + \beta)] \tag{1.2}$$

$$|x| > 0 \qquad c, \alpha > 0 \text{ and } \beta \emptyset 0$$

If $\beta = 0$, then (1.2) reduces to Inverted normal p.d.f given as:

$$g(x) = c/(2\pi\alpha)^{\frac{1}{2}}(1/x^2) \exp[-\tfrac{1}{2}(1-c/x)^2] \tag{1.3}$$

$$\text{where} \quad -\infty < x < +\infty, \quad c, \alpha > 0$$

The graphs of (1.3) for different values of α are shown in Figure (1.1).

(2) The p.d.f is bimodel and the modes are at

$$2c[1 \pm (1+8\alpha)^{2}]^{-1}$$

The p.d.f reduces to unimodal if α is small i.e. for $\alpha < 0.35$.

(3) The median is given by c.

(4) The distribution function is given by

$$G(x) = \Phi[(1-c/\alpha)/(\alpha)^{\frac{1}{2}}]$$

The p.d.f in (1.3) is related with normal distribution by

$$G(x) = N(1/x, 1/c, \alpha/c^2)$$

(5) Since mean and higher moments of p.d.f do not exist, asymptotic expressions for moments are derived using steepest descent method

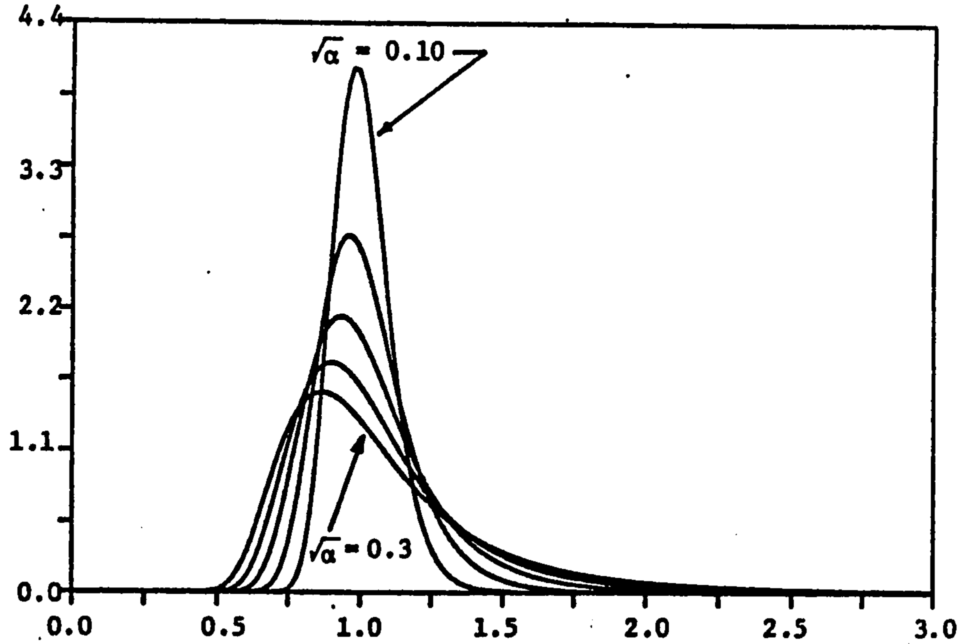

Fig. 1.1 The inverted normal probability function.

[17]. The rth non-central moment is

$$\mu'_r = c^r[1 + \sum_{m=1}^{\infty} (2m+r-1)! \quad \alpha^m/(2m)]$$

(6) The renewal rate function is approximately given by the following relationship [17]:

$$h(x) = \sum_{i=1}^{n} [nc \exp[-\tfrac{1}{2}(1-nc/x)^2(n/\alpha)]] \;/\; [(2\pi\alpha/n)^{\frac{1}{2}} x^2]$$

1.2 Negative Moments

Definition:

Let X be a positive random variable defined on the probability space (Ω, S, P). The nth negative moment of X is the expected value of $(1/X)^n$, where n is the positive integer.

Chao and Srawderman method:

In view of the applications of the negative moments Chao and Strawderman [23] proposed a method to compute the negative moments of the positive random variables. We give a summary of the method.

Definition:

Let X be a random variable defined on the probability space(Ω, S, P). Suppose that $X+A > \delta > 0$ a.s. [P]. we define the probability generating function of $(X+A-1)$ as

$$g_1(t) = E(t^{X+A-1}) \quad \text{for } 0 < t \leq 1.$$

Inductively, $g_{k+1}(t) = (1/t) \int_0^t g_k(u)du$ for $k = 1,2,3,\ldots$

Theorem: For $0 < t \leq 1$

$$E[(x+A)^{-k}\, t^{X+A}] = \int_0^t g_k(u)\, du = t\, g_{k+1}(t)$$

1.3 Bayesian Analysis

Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from the density $f(x,\theta)$ where θ is the unknown parameter. Here θ may be a vector quantity. To make an inference about θ, we have two types of methods to employ:

(1) Methods Based on Sampling Theory

In this approach we assume that θ is fixed, though unknown to us and the random sample comes from some density $f(x,\theta)$. Parametric methods of estimation like moments method, maximum likelihood method etc. belongs to this category are frequently used for estimating parameters in $f(x,\theta)$.

(2) Methods Based on Bayesian Statistics

In the Bayesian approach, we assume, in addition to the assumption that our random sample comes from a density $f(x,\theta)$, that the unknown parameter θ is the value of some random variable, say Θ. We are interested in estimating some function of θ,say $\tau(\theta)$. Assume that the density of Θ, $g(\theta)$ is known and contains no unknown parameters. The distribution of Θ is called a prior distribution of Θ. The conditional density of Θ given the sample values $x_1, \ldots, x_n$ denoted by $f(\theta | x_1, \ldots, x_n)$, is called the posterior distribution of Θ.

The Bayesian analysis often necessitates the use of inverted distributions [24,25].

1.4 OBJECTIVES:

The main objective of this research is the study of some inverted probability distributions. Specifically the proposed study will include;

(1) Derivation of inverted probability distributions corresponding to the Pearson system using reciprocal transformation.

(2) Study of a differential equation generating the inverted class of distributions.

(3) To investigate the mathematical and statistical properties of the following selected distributions:

(a) inverted inverse Gaussian

(b) inverted gamma

(c) inverted Weibull

(d) inverted Burr

First two distributions can be derived from the differential equation characterizing the inverted class of Pearson distributions whereas inverted Weibull and inverted Burr can not be derived directly from this differential equation and will be studied due to the importance of these distributions.

(4) To obtain some characterizations for the inverted probability densities.

(5) To study the nature of the hazard functions and to compare them with other well known hazard functions.

2. GENERALISED PEARSON SYSTEM OF PROBABILITY DISTRIBUTIONS

2.1 Pearson System

The Pearson system which comprises unimodel probability density functions satisfies the differential equation of the form [26,27]

$$df = - \frac{(y+a)\ f}{by^2+cy+d}\, dy \tag{2.1}$$

The family of distributions generated by the differential equation in (2.1) obey the following conditions.

(a) Every member of the family is unimodel.

(b) Has a smooth contact with the horizontal-axis at extremities,so that df/dy vanishes when f=0.

The standard Pearson distributions derived from (2.1) are shown in Table 2.1.

2.2 Inverted Class of Distributions

Inverted distributions arise in various engineering phenomena. We may derive a differential equation which generates the reciprocal transformations of the Pearson system. It should be noted that the inverted Pearson system which comprises the probability density functions corresponding to the reciprocal transform of the Pearson system includes some multimodel probability density functions as well. We call the combination of Pearson and inverted Pearson system as *the* *generalised Pearson class of probability models.*

Cobb[28] discussed a differential equation of the form

$$df(y) = [g(y)/h(y)]\, dy, \qquad (2.2)$$

$h(y) > 0$ for all y in the domain of f.

$g(y)$ and $h(y)$ are polynomial functions such that the degree of $h(y)$ is one higher than the degree of the polynomial $g(y)$. The three types of probability distributions discussed in [28] are:

1. Normal type
2. Gamma type
3. Beta type

and under certain admissible conditions on $g(y)$ and $h(y)$ are named as Catastrophy probability models. We may call $g(y)$ to be a shape function because the modes and antimodes of $f(y)$ depend upon $g(y)$.

The probability density functions which possess more than one modes or relative maxima arise from time to time in all the sciences. The generalized Pearsonian system provides a few multimodel probability distributions when the domain belongs to the whole real line or the domain includes zero. The p.d.f of inverted normal distribution is

$$f(x,c,a) = K/x^2\ )\ \exp[-1/2a(1-c/x)^2] \qquad (2.3)$$

where $K = c/(2a\pi)^{\frac{1}{2}}$ $\quad c,a > 0$ and $|x| > 0$

is a bimodel distribution.

Now, we discuss a differential equation which can be obtained from (2.1), where the degree of numerator polynomial is two and that of denominator is three. Corresponding to the Pearson differential

Table 2.1 Pearson system of probability distributions

Name of distribution	Probability density function
Normal	$(2\pi\sigma)^{-1/2} \exp[-(y-\mu)^2/(2\sigma^2)]$ $-\infty < y,\mu < \infty, \quad \sigma > 0$
Beta distribution(type i)	$[B(a,b)]^{-1}\ y^{a-1}\ (1-y)^{b-1}$ $0 < y < 1, \quad a,b > 0$
Beta distribution(type ii)	$[aB(1/2,m+1]^{-1}\ (1-y^2/a^2),$ $-a < y < a,$
Gamma distribution(type iii)	$\lambda^r/\Gamma(r)\ \ y^{r-1}\ \ \exp(-\lambda y)$ $\lambda,r > 0$ and $0 < y < \infty$
Type iv	$K(1+y^2/a^2) \exp[-\nu\tan^{-1}(y/a)],$ $m > 1/2$
inverse Gaussian(type v)	$(\lambda/(2\pi))^{1/2}\ y^{-3/2}\ \exp[-\lambda/(2\mu^2y)(y-\mu)^2],$ $y,\lambda,\mu > 0$
Beta distribution(type vi)	$1/B(p,q)\ \ y^{p-1}/(1+y)^{p+q}$ $0 < y < \infty \quad p,q > 0,$
Student t-distribution (type viii)	$[aB(1/2,m-1/2]^{-1}\ (1+y^2/a^2)^{-m}$ $-\infty < y < +\infty$

equation

$$d[\log f(y)] = - \frac{(y + a)}{d + cy + by^2} \, dy \qquad (2.4)$$

which generates the Pearson class of probability distributions, we have the differential equation

$$\frac{d}{dx}[\log g(x)] = - \frac{a_2 x^2 + a_1 x + a_0}{x(B_0 x^2 + B_1 x + B_2)} \qquad (2.5)$$

which generates the *Inverted Pearson System* and where the coefficients a_0, a_1, a_2 are given as

$$a_0 = 2B_2 - 1$$
$$a_1 = 2B_2 - a$$
$$a_2 = 2B_0$$

The differential equation in (2.5) is derived as follows:

let $Y = 1/X \quad X \neq 0$,then

$$g(x) = - x^{-2} \quad f(1/x), \text{ and}$$

$$dg/dx = 1/x^4 \quad f'(1/x) + (2/x^3)\, f(1/x)$$

$$= g(x)(1/x)\ [(1+ax)/(B_0x^2+B_1x+B_2)] - 2] \qquad (2.6)$$

which on simplification gives (2.5). If $a_2 = 0$, then the differential equation (2.5) can represent the Pearson differential equation.

Now we may generate distributions from (2.5) which will form inverted class of distributions. The form of solutions of (2.5) depends upon the nature of the roots of the equation

$$B_0 X^2 + B_1 X + B_2 = 0 \tag{2.7}$$

2.3 Inverted Probability Functions:

In this section, we derive the inverted distributions corresponding to different roots of the quadratic equation (2.7).

2.3.1 Inverted Normal Distribution:

If $B_1 = B_2 = 0$ in (2.7) then (2.6) takes the form

$$d/dx[\log g(x)] = (1/x) \quad (1 + ax)/(B_0 x^2) - 2/x$$

$$\text{or } \log g(x) = \log(c/x^2) - [1/(2B_0)](a + x^{-1})^2 + [a^2/(2B_0)],$$

$$g(x) = (K/x^2) \exp[-1/(2d) \ (1-c/x)^2] \tag{2.8}$$

where $a > 0$, $B_0 > 0$, $|x| > 0$, $c = -(1/a)$ $d = B_0/a^2$, and K is the normalization constant. The density function in (2.8) is the Inverted normal probability density function.

2.3.2 Inverted Type 1 (Inverted Beta Distribution)

This type corresponds to the case when both roots of (2.7) are real and of opposit sign. Let the roots be denoted by b_1 and b_2.Equation (2.5) can be written as

$$d/dx[\log g(x)] = (1+ax)/[-B_0 \ (x-b_1)(b_2-x) \ x \ ] - (2/x)$$

$$\text{with } \quad b_1 < 0 < b_2$$

$$=(b_1 b_2 B_0 x)^{-1} - (1+ab_1)[B_0 b_1 \ (b_2-b_1)(x-b_1)]^{-1}$$

$$- (1+ab_2) \ [B_2 b_2 (b_2-b_1)(b_2 - x)]^{-1} - (2/x)$$

$$\text{where } b_1 < x < b_2,$$

which after integration and some manipulations yields

$$g(x) = K_1 (x-b_1)^p (b_2-x)^q x^{-p-q-2}$$

$$= K_1 [1-(b_1/x)]^p [(b_2/x) - 1]^q x^{-2} \qquad (2.9)$$

where $p = -(1+ab_1)/[b_1 B_0(b_2-b_1)]$, and

$$q = (1+ab_2)/[b_2 B_0 (b_2 - b_1)]$$

which is the general form of the Inverted Beta probability function. The range of x is $0< b_1 < x < b_2$. To derive the standard form of the Inverted Beta distribution we make the following transformation

$$w = [x(b_2-b_1)]/[b_2(x-b_1)]$$

Where $1 < w < \infty$

The jacobian of the transformation turns out to be

$$dx/dw = [b_1 b_2(b_1- b_2)]/[b_2 w-(b_2 - b_1)]^2$$

Substituting these values in equation (2.9) we obtain

$$g(x) = C (w-1)^q w^{-p-q-2} \qquad 1 < w < \infty \qquad (2.10)$$

where C is the normalization constant. The function (2.10) is the standard form of the inverted type i.

2.3.3 Inverted Type ii:

If $p = q = m$ say , and $-b_1 = b_2$ in (2.9) then we obtain

$$g(x) = K [1-(a/x)]^m [(a/x)+1]^m (1/x^2)$$

$$= K [1-(a^2/x^2)]^m (1/x^2) \quad \text{where } -a \le x \le a$$

2.3.4 Inverted Type iii

This type corresponds to the case when $B_2 = 0$ but $B_1 \neq 0$ in

(2.7). With these values of B_1 and B_2, equation (2.5) can be written as

$$g'(x)/g(x) = (1+ax) / [x(B_0x^2 + B_1x)] - (2/x)$$
$$= (aB_1 - B_0) / (xB_1^2) + (B_1x^2)^{-1}$$
$$+ (B_0^2-aB_0B_1) (B_0B_1^2x+B_1^3)^{-1} - (2/x)$$

$$\text{or} \quad g(x) = (K/x^2) \quad (B_0+B_1/x)^{m-1} \exp[-(B_1x)^{-1}] \qquad (2.11)$$

where $m = (B_0-aB_1/B_1^2) + 1$

and K is the normalization constant.

$g(x)$ is the general form of the Inverted Gamma function. If $B_1 > 0$ we take the range of X as $X > -(B_1/B_0)$. If $B_1 < 0$ then the range of x is $x < (B_1/B_0)$.

Using the transformation

$X = B_1w(1-B_0w)^{-1}$ for which the jacobian is $B_1(1-B_0x)^{-2}$ and

substituting these values in (2.11), we obtain

$$g(w) = K_1 \, (1/w^{m+1}) \exp(-\lambda/w)$$

where $\lambda = 1/B_1^2$, $K_1 = K \exp(B_0/B_1^2)/B_1$, and $0 < w < \infty$.

Integrating over the domain of g(x) and equating it to one we find the value of the normalization constant. Finally we obtain

$$g(w) = \lambda^m \, (1/\Gamma(m)) \, (1/w^{m+1}) \exp(-\lambda/w) \qquad \text{for } w > 0,$$

The pdf, g(w) is the standard Inverted Gamma density function.

2.3.5 Inverted Type v :

This type corresponds to the case when $B_0 x^2 + B_1 x + B_2 = 0$ is a perfect square. Equation (2.5) becomes

$$g'(x)/g(x) = (1+ax) / [xB_0(x+c_1)^2] - (2/x)$$

$$(d/dx)\log g(x) = (xB_0c_1{}^2)^{-1} - [B_0c_1{}^2(x+c_1)]^{-1}$$

$$+ (ac_1-1)[c_1B_0(x+c_1)^2]^{-1} - (2/x)$$

Integrating and solving for g(x) we obtain

$$g(x) = (k/x^2)[(x+c_1)/x]^{-(1/(B_0c_1{}^2))}$$
$$\exp[(1-ac_1)/[B_0c_1(x+c_1)]] \qquad (2.12)$$

If $(1-ac_1)/(B_0c_1) < 0$ then $x > -c_1$

If $(1-ac_1)/(B_0c_1) > 0$ then $x < -c_1$

If $a = 1/c_1$ then we have the special case

$$g(x) = (k/x^2)\ [(x+c_1)/x]^{-(1/(B_0c_1{}^2))}$$

which is the Inverted Type Viii and iX according to as $B_0 > 0$ or $B_0 < 0$.

2.3.6 Inverted Type vi:

This type corresponds to the case when the roots of (2.7) are real and of the same sign but not equal. Let α_1 and α_2 be the roots of (2.7) such that $\alpha_1, \alpha_2 > 0$ and $\alpha_1 \neq \alpha_2$. Then equation (2.5) can be written as

$$d/dx[\log g(x)] = (1+ax)/[x(x-\alpha_1)(x-\alpha_2)] - (2/x)$$

$$= (\alpha_1\alpha_2 x)^{-1} - (1+a\alpha_2)\ [\alpha_2(\alpha_1-\alpha_2)(x-\alpha_1)]^{-1}$$

$$+ (1+a\alpha_1)[\alpha_1(\alpha_1-\alpha_2)\ (x-\alpha_2)]^{-1} - (2/x)$$

The resulting probability density function is

$$g(x) = K\ (x-\alpha_2)^m\ [x^{m-n+2}(x-\alpha_1)^n]^{-1}.$$

If $\alpha_1 > \alpha_2$ then $m,n > 0$ and $0 < \alpha_1 < x < \infty$

where $m = (1+a\alpha_1)/[\alpha_1(\alpha_1-\alpha_2)]$

and $n = (1+a\alpha_2)/[\alpha_2(\alpha_1-\alpha_2)]$.

2.3.7 Inverted Type vii:(Inverted Student t-Distribution)

This probability density function corresponds to the case when

$a = B_1 = 0\ , B_0 > 0\ , B_2 > 0$

so we have

$$d/dx[\log g(x)] = 1/[x(B_0x^2+B_2)] - (2/x)$$

$$= (B_2x)^{-1} - (B_0/B_2)x(B_0x^2+B_2)^{-1} - (2/x),$$

which have the solution

$$g(x) = K\ \ x^{(1/B_2)-2}\ (B_0x^2+B_2)^{-(2B_2)^{-1}}$$

or

$$g(x) = K_1\ [1+1/(a^2x^2)]^{-m}\ (1/x^2), \tag{2.13}$$

where $|x| > 0$, $K_1 = K/(B_0{}^{1/2B_2})$, $m = (1/2B_2)$ and $a^2 = B_0/B_2$.

Integrating g(x) over its domain and substituting the value of the normalization constant we finally obtain

$$g(x) = \frac{[1+1/(a^2x^2)]^{-m}}{a\ B(1/2,m-1/2)\ x^2} \quad \text{for } |x| > 0,$$

which is the Inverted Student t-distribution.

2.3.8 Inverted Type iv:

If the equation (2.7) does not have real roots, then we write

$B_0x^2 + B_1x + B_2 = B_2x^2[[(1/x) + B_1/(2B_2)^2] + (B_0/B_2) - B_1^2/(4B_2^2)]$

Therefore, equation (2.5) takes the form

$d/dx[\log g(x)] = 1/[B_2x^3(x+\gamma)^2 + \delta^2] - (2/x)$,

where $\gamma = B_1/(2B_0)$ and $\delta^2 = B_0/B_2 - B_1^2/(4B_2^2)$

Integrating and simplifying we obtain

$g(x) = K[(x^{-1}+\gamma)^{-2}+\delta^2]^{(1/2B_2)}\ (1/x^2)\ \exp[-\gamma/(B_2\delta)\ \arctan((x)+\gamma)/\delta]$,

$|x| > 0.$

2.4 RELATIONSHIP BETWEEN THE MOMENTS AND THE PARAMETERS:

The differential equation in (2.5) contains four parameters B_0, B_1, B_2 and a. We can express these parameters in terms of the first five moments . Equation (2.5) can be written as

$x^{n+1}(B_0x^2+B_1x+B_2)(dg/dx) + [2B_0x^2 + (2B_2-a)x + (2B_2-1)]x^ng\ (x) = 0$

Integrating the above equation between $-\infty$ to $+\infty$,assuming that the integrals exist and $\lim_{x\to+\infty}x^{n+1}g(x) = 0$, $\lim_{x\to-\infty} x^{n+1}\ g(x) = 0$.

$$(n+1)B_0\mu'_{n+2} + nB_1\mu'_{n+1} - (1-n)B_2\mu_n' + \mu_n' + a\mu'_{n+1} = 0$$

Substituting n = 0,1,2,3 and noting that $\mu'_0 = 1$ and taking $\mu'_1 = 0$ we obtain

$$\mu_2B_0 - B_2 + 1 = 0,$$
$$2\mu_3B_0 + \mu_2B_1 + \mu_2a = 0,$$
$$3\mu_4B_0 + 2\mu_3B_1 + \mu_2B_2 + \mu_3a + \mu_2 = 0,$$
$$4\mu_5B_0 + 3\mu_4B_1 + 2\mu_3B_2 + \mu_4a + \mu_3 = 0.$$

Solving the above system of four equations in four unknowns , the formulas for B_0, B_1, B_2, and a are found as

$$B_0 = (3\mu_3^2-2\mu_2\mu_4) \;/\; (3\mu_2\mu_4+\mu_4\mu_2^2- 4\mu_3\mu_5-2\mu_2\mu_3^2),$$

$$B_1= (6\mu_3^4+8\mu_2^2\mu_5+\mu_3\mu_2^3-13\mu_2\mu_3\mu_4) \;/D,$$

$$a=(17\mu_2\mu_3\mu_4-\mu_3\mu_2^3-12\mu_3^3-8\mu_5\mu_2^2)/D,$$

and

$$B_2=(\mu_3^2\mu_2-\mu_4\mu_2^2+3\mu_2\mu_4-4\mu_3\mu_5)/(3\,\mu_2\mu_4+\mu_4\mu_2^2-4\mu_3\mu_5-2\mu_2\mu_3^2),$$

where $D=3\mu_4\mu_2^2+\mu_4\mu_2^3-4\mu_2\mu_3\mu_5-2\mu_2^2\,\mu_3^2$.

2.5 SOME OTHER USEFUL INVERTED DISTRIBUTIONS

The inverted Burr and inverted Weibull probability models do not belong to the inverted Pearson system. Since in authors opinion these are important models in reliability study, therefore these will be investigated in chapters 4 and 6.

3. INVERTED INVERSE GAUSSIAN DISTRIBUTION

The inverse Gaussian class of distributions was derived by Wald[29] and was studied by Tweedie[30], who observed that there exist an inverse relationship between the cumulant generating function of these distributions and the cumulant generating function of the Gaussian distributions. Due to this relationship, this class of distributions was given the name "Inverse Gaussian" distribution. Wasan[31] called these distributions as the first passage time distributions of Brownian motion with positive drift. Many other authors have also studied this probability model rigorously.

3.1 PHYSICAL INTERPRETATION OF THE INVERTED INVERSE GAUSSIAN DISTRIBUTION FROM THE BROWNIAN MOTION WITH POSITIVE DRIFT:

Suppose a particle moving along a straight line tends to move with a uniform velocity v. Suppose also that, the particle is subject to a linear Brownian motion which causes it to take a variable amount of time to cover a fixed distance d. It has been shown [32] that the time y required to cover the distance d is a random variable with probability density function

$$f(y) = d/(2\pi\beta y^3)^{1/2} \exp[-(d-vy)^2/(2\beta y)], \qquad (3.1)$$

where $y > 0$,

where β is the diffusion constant. On substituting $v = d/\mu$ and $\beta = d^2/\lambda$ in (3.1) we obtain, in different notations the inverse Gaussian probability density function:

$$f(y,\lambda,\mu)=[\lambda/(2\pi y^3]^{1/2} \exp[-\lambda(y-\mu)^2/(2\mu^2 y)], \quad y,\mu,\lambda > 0. \tag{3.2}$$

Figure (3.1) shows the graph of this probability density.

The average speed X of the moving particle subject to the linear Brownian motion as explained above is the reciprocal of the time Y required to cover the distance d. Thus the average speed is a random variable with probability density function which corresponds to the reciprocal transformation $X = Y^{-1}$. The probability density of X is

$$g(x,\lambda,\mu) = (\lambda/2\pi)^{1/2} x^{-1/2} \exp[-\lambda(x-\mu^{-1})^2/(2x)] \quad x,\mu,\lambda > 0. \tag{3.3}$$

The probability density function in (3.3) is the inverted inverse Gaussian density which has been named as *Random Walk Distribution* by [26,31,33]. The following results have been found in literature about the inverted inverse Gaussian distribution [26].

(1) The cumulant generating function of X is

$$\psi(t) = \phi(1 - (1 + 2t\lambda^{-1})^{1/2}) - (1/2) \ln(1 + 2t\lambda^{-1})$$

and the first two cumulants are

$$k_1(x) = \mu^{-1} + \lambda^{-1}$$

$$k_2(x) = (\lambda\mu)^{-1} + 2\lambda^{-2}$$

(2) The mode of the density function of X is

$$X_{mode} = (1/\mu)[(1 + (4\phi^2)^{-1})^{1/2} - (1/2\phi)], \quad \text{where } \phi = \lambda/\mu.$$

(3) The density function has two points of infelection at values of x

satisfying the equations

$$u^4 + 2u^3 + u = (\lambda/\mu)^2 + 1/4,$$

where $u = 1/2 + 1/2(\lambda/\mu)(x/\mu - \mu/x)$.

(4) For $\mu = 1$ we have a relation between the moments of the inverse Gaussian distribution and the inverted inverse Gaussian distribution as

$$\mu'_{-r} = \mu'_{r+1}$$

(5) The first three moments about zero of the inverted inverse Gaussian distribution are given as

$$\mu_1' = 1/\mu + 1/\lambda,$$

$$\mu_2' = 1/\mu + 3(\lambda\mu)^{-1} + 3\lambda^{-2},$$

$$\mu_3' = 1/\mu^{-3} + 6(\lambda\mu^2)^{-1} + 15(\mu\lambda^2)^{-1} + 15\lambda^{-3}.$$

(6) The Inverted Inverse Gaussian distribution has shown to be infinitely divisible [34].

In this chapter, we shall explore some other mathematical and statistical properties of Inverted Inverse Gaussian probability model and prove some characterization results.

The Inverted Inverse Gaussian distribution is unimodel and the mode is given by the equation

$$X_{mode} = (1/\lambda)[(4 + \mu^{-2}\lambda^2)^{1/2} - 2]$$

Figure 3.2 show the graphs of inverted inverse Gaussian density functions for different values of the shape parameter λ.

3.2 MOMENT GENERATING FUNCTION AND THE CHARACTERISTIC FUNCTION:

The moments of the distribution has been derived in literature[26] by direct integration. These moments can also be calculated by using the moment generating function.

Let X be a random variable having an Inverted Inverse Gaussian distribution with probability density as in equation (3.3). Then by definition

$$M_X(t) = E[\exp(tX)] = \int_{-\infty}^{+\infty} \exp(tx) g(x,\lambda,\mu)\, dx$$

$$M_X(t) = (\lambda/2\pi)^{1/2} \exp(\lambda\mu^{-2}) \int_{-\infty}^{+\infty} \exp[-(\lambda/2-t)x - \lambda\mu^{-2}/(2x)]\ x^{-1/2} dx \qquad (3.4)$$

Let $y = (\lambda/2 - t)x$,

$dx = dy/(\lambda/2 - t)$.

Substituting these values in the above integral and letting

$$Z_t = [\lambda\mu^{-2}(\lambda - 2t)]^{1/2}$$

we obtain

$$M_X(t) = K[(1/2)(Z_t/2)^{-1/2}] \int_0^{\infty} \exp(-y - Z_t^2/4y) y^{1/2-1}\, dy \qquad (3.5)$$

where $K = (\lambda/2\pi)^{1/2}\ 2\exp(\lambda/\mu)(\lambda/2 - t)^{-1/2}(Z_t/2)^{1/2}$.

To evaluate this integral we use a modified Bessal function of the

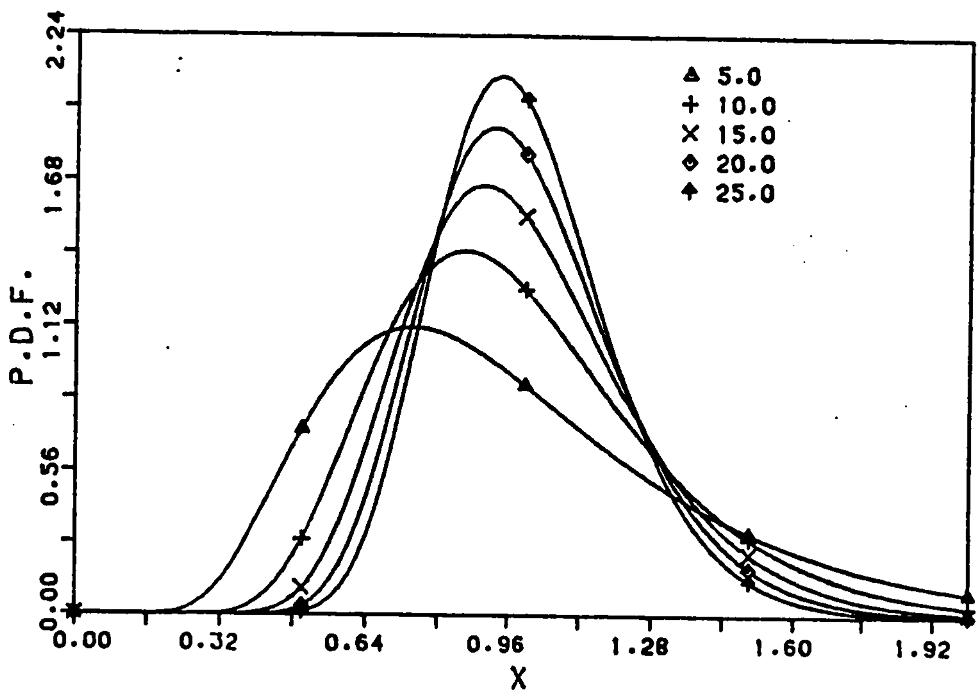

Fig. 3.1 The inverse Gaussian probability density function.

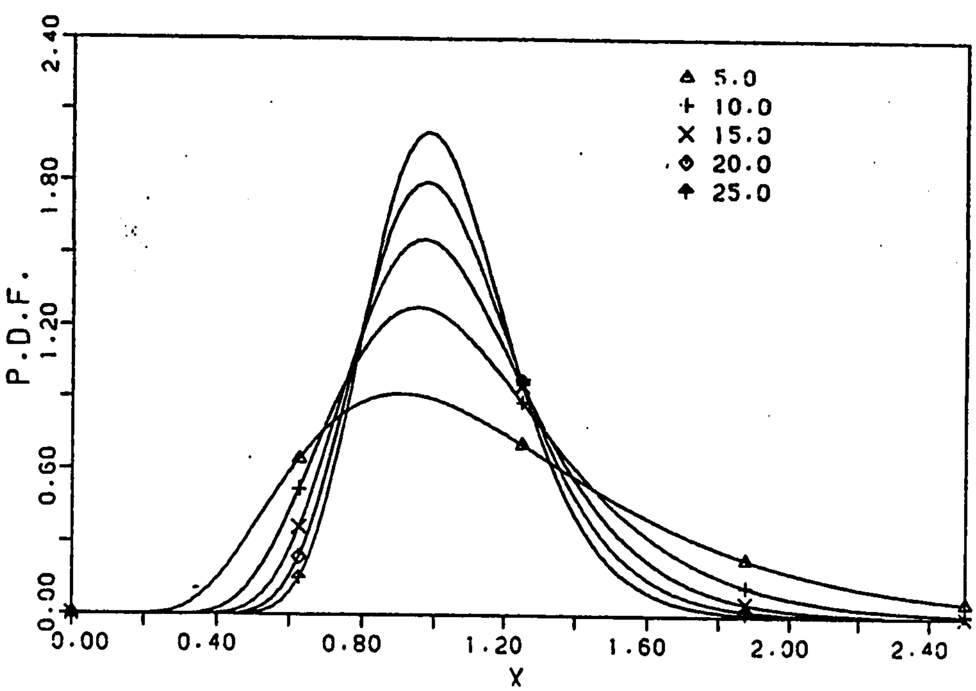

Fig. 3.2 The inverted inverse Gaussian probability function.

second kind [p.183,35]

$$\kappa_{+\nu}(Z_t) = (1/2)(Z_t/2)^{\nu}\,{}_0\!\int^{\infty}\exp[-t-Z_t^{2}/(4t)]\ t^{-\nu-1}dt.$$

The solution of which for $\nu = 1/2$ is [p.80,35]

$$\kappa_{1/2}(Z_t) = \exp(-Z_t)[\pi/(2Z_t)]^{1/2}.$$

Therefore, the equation (3.5) becomes

$$M_x(t) = \exp[\lambda/\mu - \lambda/\mu(1 - 2t/\lambda)^{1/2}]\ (1-2t/\lambda)^{-1/2}. \tag{3.6}$$

Using the definition $\mu_n' = (d^n/dt^n)M_x(t)|_{t=0}$,

we obtain the moments about zero .The first three moments yield to be

$$\mu_1' = 1/\mu + 1/\lambda \tag{3.7}$$

$$\mu_2' = \mu^{-2} + 3/(\mu\lambda) + 3/\lambda^2$$

$$\mu_3' = \mu^{-3} + 6\mu^{-2}/\lambda + 15\mu^{-1}\lambda^{-2} + 15\lambda^{-3}$$

and $$\mathrm{Var}(X) = (\mu\lambda)^{-1} + 2\lambda^{-2}. \tag{3.8}$$

NOTE:

(1) The negative moments of the inverse Gaussian distribution are the positive moments of the inverted inverse Gaussian distribution.

(2) The positive integral moments about zero may be derived by direct integration and by making use of the modified Bessel function of the second kind [p.183,35].

Definition: Let μ be a finite measure on B(R). The characteristic function of μ is the mapping from R to C given by

$$h(t) = \int_R \exp(itx)\, d\mu(x) \quad t \in R.$$

Characteristic functions are appropriate in the study of sums of independent random variables. Using a method analogous to the moment generating function, the characteristic function of the inverted inverse Gaussian distribution turns out to be

$$h(t) = \exp[\lambda\mu^{-1}(1-(1-2it/\lambda)^{1/2})](1-2it/\lambda)^{-1/2}. \qquad (3.9)$$

3.3 PARAMETRIC POINT ESTIMATIONS:

(a) Maximum Likelihood Estimates Of The Parameters:

Let $X_1, X_2, \ldots, X_n$ be a random sample from a population having probability density function (3.3). The maximum likelihood estimates of μ and $1/\lambda$ are $(1/n)\sum_{i=1}^{n} x_i^{-1}$ and $\overline{X}-H_n$ respectively,

where $[H_n]^{-1} = (1/n)\sum_{i=1}^{n} x_i^{-1}$.

Proof: The likelihood function is

$$L(\lambda,\mu) = C(X)\, \lambda^{n/2} \exp[-(\lambda/2)\sum_{i=1}^{n} (x_i-1/\mu)^2/x_i] \qquad (3.10)$$

where $C(X) = (2\pi)^{-n/2} \prod_{i=1}^{n} x_i^{-1/2}$

The estimating equations from the likelihood function are

$$1/\hat{\lambda} = \overline{X} + (H_n\hat{\mu}^2)^{-1} - 2/\hat{\mu} \tag{3.11}$$

$$\hat{\mu} = 1/H_n \tag{3.12}$$

Substituting (3.12) in (3.11), We get $1/\hat{\lambda} = (\overline{X} - H_n)$

(b) Variance-Covariance Matrix:

It is seen that the first and second order derivatives of $\ln L(\lambda,\mu)$ exist for $\mu,\lambda \neq 0$. Since $\ln L$ is a monotonically increasing function of L, the first and second order deritives of L also exist. The measurment space $[0 < x < +\infty]$ is independent of λ and μ. Hence the regularity conditions are satisfied for the inverted inverse Gaussian distribution. For the logarithm of the likelihood function,

$$\ln L = (n/2)\ln\lambda - (n/2)\ln(2\pi) - (\lambda/2)\sum_{i=1}^{n} (x_i-\mu^{-1})^2(x_i)^{-1} - (1/2)\sum_{i=1}^{n} \ln x_i$$

we have $-E(\partial^2 \ln L/\partial\lambda^2) = n/(2\lambda^2)$,

$$-E(\partial^2 \ln L/\partial\mu^2) = n\lambda/\mu^3,$$

and $$\partial^2 \ln L/\partial\lambda\partial\mu = -n/\mu^2 + (1/\mu^3)\sum_{i=1}^{n} x_i^{-1}.$$

Since $E(1/X) = \mu$, we have

$-E(\partial^2 \ln L/\partial\lambda\partial\mu) = 0.$

The variance-covariance matrix V is

$$V = \begin{bmatrix} n\lambda/\mu^3 & 0 \\ 0 & n/2\lambda^2 \end{bmatrix}^{-1} = \begin{bmatrix} \mu^3/n\lambda & 0 \\ 0 & 2\lambda^2/n \end{bmatrix}.$$

$Cov(\hat{\lambda},\hat{\mu}) = 0$

$Var(\hat{\lambda}) = 2\hat{\lambda}^2/n$

$Var(\hat{\mu}) = \hat{\mu}^3/n\hat{\lambda}$

In both the cases , variance is inversely propotional to the sample size n and as such, $\hat{\lambda}$ and $\hat{\mu}$ are consistent estimators of λ and μ respectively.

(c) Sufficient Statistic:

Let $X_1,X_2,\ldots,X_n$ denote a random sample from the distribution (3.3). Let

$$Y_1 = (1/n)\sum_{i=1}^{n} X_i^{-1}$$

than Y_1 is a sufficient statistic for μ when λ is known.

Proof: The probability density of the statistic Y_1 is inverse Gaussian [36] with parameters $n\lambda$ and μ i.e.

$$f(y_1,n\lambda,\mu) = [n\lambda/(2\pi)]^{1/2}\ y_1^{-3/2}\ \exp[-n\lambda/2\mu y_1^{2}\ (y_1 - \mu)^2]$$

$$\text{for } y_1,n\lambda,\mu > 0$$

The joint probability density function of $x_1,\ldots,x_n$ may be written as

$$(\lambda/(2\pi))^{n/2}\ (\prod_{i=1}^{n} x_i^{-1/2})\ \exp((-\lambda/2)\sum_{i=1}^{n}\ [(x_i - 1/\mu)^2(x_i)^{-1}])$$

$$= (n\lambda/(2\pi))^{1/2} y_1^{-3/2}\ \exp[-n\lambda/(2y_1\mu^2)\ (y_1 - \mu)^2](n)^{-1/2}$$

$$[\lambda/(2\pi)]^{(n-1)/2}\ y_1^{3/2}(\prod_{i=1}^{n} x_i^{-1/2})\exp((-\lambda/2)$$

$$\sum_{i=1}^{n}\ [x_i + n\lambda/(2y_i)].$$

In accordance with the Fisher-Neyman criterion for sufficiency [37,p.216] , Y_1 is a sufficient statistic for μ.

For joint sufficiency we give the following theorem.

Theorem 3.3.1 Let $X_1,\dots,X_n$ be a random sample from the population (3.3).Then $(1/H_n,\overline{X}-H_n)$ is a jointly sufficient statistic for (μ,λ).

Proof: The joint p.d.f of $X_1,\dots,X_1$ can be written as

$$g(x_1,\dots x_n) = [\lambda/(2\pi)]^{n/2}(\prod_{i=1}^{n} x_i^{-1/2})\exp(-n\lambda/(2\mu^2 H_n^{-1}) (H_n^{-1} - \mu)^2)$$

$$\exp[(-n\lambda/2)(\overline{X}-H_n)]$$

$$= K (\prod_{i=1}^{n} x_i^{-1/2}) \quad h(1/n \sum_{i=1}^{n} x_i^{-1}, \overline{X} - H_n,\mu, \lambda)$$

where $K = (\lambda/2\pi)^{n/2}$ and $H_n = [(1/n)\sum_{i=1}^{n} x_i^{-1}]^{-1}$

By factorization criterion for jointly sufficiently statistics we conclude that $(H_n^{-1},\overline{X} - H_n)$ is a jointly sufficient statistics for (μ,λ).

(d) Cramer Rao Lower Bounds For The Variance Of Unbiased Estimators Of The Parameters:

Theorem 3.3.2 (Cramer Rao Inequality):[38]

Let $X_1,\dots,X_n$ be a random sample from $f(x,\theta)$. Let $T = t(x_1,\dots,x_n)$ be an unbiased estimator of $\tau(\theta)$. Then under the regularity conditions

$$\mathrm{Var}_\theta[T] \le [\tau'(\theta)]^2/[nE_\theta((\partial/\partial\theta \log f(x,\theta))^2]$$

Equality prevails iff there exist a function ,say $K(\theta,n)$ such that

$$\sum_{i=1}^{n} \partial/\partial\theta \log f(x_i,\theta) = K(\theta,n)[t(x_1,\dots,x_n) - \tau(\theta)]$$

Let $X_1,\ldots,X_n$ be a random sample of a random variable with density function

$$g(x,\lambda,\mu) = (\lambda/2\pi)^{1/2}\ x^{-1/2}\ \exp[-\lambda(x - \mu^{-1})/(2x)].$$

Take $\tau_1 = \mu$ and $\tau_2 = 1/\lambda$ and notice that the regularity conditions are satisfied. Let $T_1 = 1/H_n$ and $T_2 = \overline{X} + (\mu^2 H_n)^{-1} - (2/\mu)$ (for μ to be known).

Then $\quad E(T_1) = E(1/H_n) = \mu$

and $\quad E(T_2) = E(\overline{X}) + (1/\mu^2)E(1/H_n) - 2\mu^{-1} = 1/\lambda$

Therefore, T_1 and T_2 are unbiased estimators of τ_1 and τ_2 respectively. Also

$$\tau_1' = 1 \quad \text{and} \quad \tau_2' = -1/\lambda^2$$

$$\log g(x,\lambda,\mu) = (1/2)\log[(\lambda/(2\pi)] - (1/2)\log x - [\lambda/(2x)](x-1/\mu)^2$$

$$\partial/\partial\lambda[\log g(x,\lambda,\mu)] = 1/(2\lambda) - (2x)^{-1}(x - 1/\mu)^2$$

$$E_\lambda[(\partial/\partial\lambda \log g(x,\lambda,\mu))^2] = E_\lambda[1/(2\lambda) - (X - 1/\mu)^2/2X\ ]^2$$

$$= \mathrm{Var}_\lambda[(1/2)(X + (\mu^2 X)^{-1})]$$

$$= (\lambda + 2\mu + \lambda\mu^2)/(2\lambda^2\mu)$$

Hence, by theorem (3.3.2), for μ known the Cramer Rao lower bound for $1/\lambda$ is

$$\mathrm{Var}_{1/\lambda}[T_2] \le (2\mu)[n\lambda^2(\lambda+2\mu+\lambda\mu^2)]^{-1} \qquad (3.13)$$

$$\partial/\partial\mu[\log g(x,\lambda,\mu)] = -\lambda/\mu^2 + \lambda/(x\mu^3)$$

and $\quad E_\mu[(\partial/\partial\mu \log g(x,\lambda,\mu))^2] = (\lambda^2)/\mu^6\ \mathrm{Var}(1/X)$

$$= (\lambda^2/\mu^6)\ (\mu^3/\lambda) = \lambda/\mu^3$$

Similarly the Cramer Rao lower bound for the estimator T_1 is given by

$$\mathrm{Var}_{\mu}[T_1] \le 1/(n\lambda/\mu^3) \qquad (3.14)$$

(e) Uniformly Minimum Variance Unbiased Estimators of The Parameters:
After some manipulations we have the relations:

$$\sum_{i=1}^{n} [\partial/\partial\lambda \log g(x,\lambda,\mu)] = - n/2[\overline{X} + 1/(\mu^2 H_n) - (2/\mu) - 1/\lambda], \qquad (3.15)$$

$$\sum_{i=1}^{n} \partial/\partial\mu \log g(x,\lambda,\mu) = (n\lambda)/\mu^3(1/H_n - \mu) \qquad (3.16)$$

From (3.15), (3.16) we conclude that $1/H_n$ and $(\overline{X} + (H_n\mu^2)^{-1} - 2/\mu)$ are UMVUBE of μ (for λ known) and $1/\lambda$ (for μ known) respectively.

3.4 STATISTICAL DISTRIBUTIONS RELATED TO INVERTED INVERSE GAUSSIAN

One of the features of this distribution is that the distribution function of the inverted inverse Gaussian distribution can be expressed in terms of the distribution function of the standard normal. To show this we prove the following result.

Theorem.3.4.1

Let X be an inverted inverse Gaussian random variable and let

$$Y = (\lambda/X)^{1/2} (X - 1/\mu)$$

Then the density function of Y is given by

$$f(y,\lambda,\mu) = [1 + y(y^2 + 4\lambda\mu^{-1})^{-1/2}] \, \phi(y) \quad -\infty < y < \infty$$

Where $\phi(y)$ is standard normal probability density function.

Proof: The transformation $Y = (\lambda/X)^{1/2}\ (X-\mu^{-1})$ is one to one and as X varies between 0 and ∞, Y varies between $-\infty$ to $+\infty$.

$$dx/dy = 2x^{3/2}\ \lambda^{-1/2}\ (x + 1/\mu)^{-1}$$

$$dx/x^{1/2} = 2x(x + 1/\mu)^{-1}\ \lambda^{-1/2}\ dy \qquad (3.17)$$

The positive root of $x^{1/2}\ y = \lambda^{1/2}(x - 1/\mu)$ for x is

$$x = [2\lambda/\mu + y^2]/(2\lambda) + y(y^2 + 4\lambda/\mu)^{1/2}/(2\lambda)$$

Also $2x/(x + \mu^{-1}) = 1 + y(y^2 + 4\lambda\mu^{-1})^{-1/2}$ (3.18)

Therefore,

$g(y,\lambda,\mu) = |J|\ f(x,\lambda,\mu)$

where J is the jacobian of the transformation.

Making use of equation (3.17) and equation (3.18),we write the density g(x) of X as

$$g(y,\lambda,\mu) = (2\pi)^{-1/2}\ \exp(-y^2/2)\ [1 + y(y^2 + 4\lambda\mu^{-1})^{-1/2}]$$

$$= \phi(y)\ [1 + y(y^2 + 4\lambda\mu^{-1})^{-1/2}] \qquad (3.19)$$

Where $\phi(y)$ is the standard normal function. We will use this result to find the distribution function of the inverted inverse Gaussian distribution.

A Nonlinear Weighted Normal:

Colloray: 3.4.2

Let X be an inverted inverse Gaussian random variable. Let

$$Y = (\lambda/X)^{\frac{1}{2}}(X - 1/\mu)$$

Then the distribution function of Y is

$$G(y) = \Phi(y) + \exp(2\lambda/\mu)\ \Phi[-(4\lambda/\mu + y^2)^{\frac{1}{2}}]$$

where $\Phi(y)$ is the cumulative distribution function of $N(0,1)$.

Proof:

To find the distribution function of inverted inverse Gaussian distribution we make use of (3.19) and write

$$G(y) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{y} \exp(-z^2/2)\,dz + (2\pi)^{-\frac{1}{2}}$$

$$\int_{-\infty}^{y} z(4\lambda/\mu + z^2)^{-1/2} \exp(-z^2/2)\,dz$$

$$G(y) = \Phi(y) + (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{y} [z^2/(4\lambda/\mu + z^2)]^{\frac{1}{2}} \exp(-z^2/2)\,dz$$

Let $R = 1/(2\pi)^{\frac{1}{2}} \int_{-\infty}^{y} (z^2/(4\lambda/\mu+z^2))^{\frac{1}{2}} \exp(-z^2/2)\,dz$

Making transformation $u = (4\lambda\mu^{-1} + z^2)^{\frac{1}{2}}$

$$du = z/(4\lambda\mu^{-1} + z^2)^{1/2}\,dz$$

Therefore, $R = \exp(2\lambda/\mu)(2\pi)^{-\frac{1}{2}} \int_{u_0}^{\infty} \exp(-u^2/2)du,$

where $u_0 = (4\lambda/\mu + y^2)$

$$= \Phi[-(4\lambda/\mu+y^2)^{\frac{1}{2}}]\exp(2\lambda/\mu) \text{ where } -\infty < y < \infty.$$

Hence

$$G(y) = \Phi(y) + \Phi[-(4\lambda/\mu + y^2)^{\frac{1}{2}}]\exp(2\lambda/\mu), \quad -\infty < y < \infty.$$

Note: We call the distribution of Y a nonlinear weighted normal because of the nonlinear weighted form of the normal density function in (3.19).

We have the following result showing the relationship of chi-square distribution and the inverted inverse Gaussian.

Theorem:3.4.3

Let X be a random variable with density function $g(x,\lambda,\mu)$.Then $(\lambda/X)(X - 1/\mu)^2$ is a chi-square random variable with one degree of freedom.

Proof:

Let $Y = (\lambda/X)(X - 1/\mu)^2$

The moment generating function of Y is

$$M_Y(t) = [\lambda/(2\pi)]^{\frac{1}{2}} \int_0^\infty x^{-1/2} \exp[t\lambda x^{-1}(x-1/\mu)^2 - \lambda\ (2x)^{-1}(x-1/\mu)^2]\ dx$$

$$= [\lambda/(2\pi)]^{\frac{1}{2}} \int_0^\infty x^{-1/2} \exp[-\lambda/(2x)(1 - 2t)^{\frac{1}{2}}x - (\mu^{-2}-2t\mu^{-2})^{\frac{1}{2}})]^2\ dx$$

$$= [\lambda/(2\pi)]^{\frac{1}{2}} \int_0^\infty g(x,\lambda(1 - 2t),(\mu^{-2}-2t\mu^{-2})^{\frac{1}{2}}/(1 - 2t)^{\frac{1}{2}}))dx$$

$$= (1 - 2t)^{-1/2} \quad \text{for } t < 1/2,$$

which is the moment generating function of the chi-square distribution with one degree of freedom. Hence $(\lambda/X)(X - \mu^{-1})^2$ is a chi-square random variable with one degree of freedom.

3.5 CHARACTERIZATIONS:

Theorem:3.5.1

Let $X_1,\ldots,X_n$ be mutually independent random variables distributed according to the inverted inverse Gaussian density $g(x,\lambda,1)$. Let $Y = \sum_{i=1}^{n} X_i$. Then the distribution of Y can be regarded as the convolution of the p.d.f $g_1(z_1,n^2\lambda,n)$ and the p.d.f $g_2(z_2/\lambda)$, where Z_1 is the inverse Gaussian random variate and Z_2 is the chi-square random variable with n degrees of freedom.

Conversely, if Y is the convolution of $\chi^2(n)$ with p.d.f $g_1(z_2/\lambda)$

and inverse Gaussian with p.d.f $g_2(z_1, n^2\lambda, n)$, then each X_i is inverted inverse Gaussian random variable with p.d.f. $g(x,\lambda,1)$.

Proof:

The characteristic function $\Psi_Y(t)$ of $Y = \sum_{i=1}^{n} X_i$ is

$$\Psi_Y(t) = \exp[n\lambda(1 - (1-2it/\lambda)^{1/2})]\ (1 - 2it/\lambda)^{-n/2} \tag{3.20}$$

$$= \Psi_{Z_1}(t)\ \Psi_{Z_2}(t) = \Psi_{Z_1+Z_2}(t)$$

Now, we identify $\exp[n\lambda(1 - 2it/\lambda)^{1/2}]$ as the characteristic function of inverse Gaussian random variate with probability density $g_1(z_1, n^2\lambda, n)$ and $(1 - 2it/\lambda)^{-n/2}$ is the characteristic function of $x^2(n)$ with probability density $g_2(z_2/\lambda)$. Hence Y is the convolution of $g_2(z_2/\lambda)$ and $g_1(z_1, n^2\lambda, n\mu)$.

The converse follows immediately from equation (3.20).

Theorem:3.5.2

Let X be a random variable having probability density function $g(x)$ and distribution function $G(x)$. If $E(1/X)$ exists and

$$\int \exp(itx)\ x^{-1}\ dG(x) = (\mu - i\rho t)^{1/2} \int \exp(itx)\ dG(x) \tag{3.21}$$

for some $\rho > 0$, $\mu > 0$ and $|t| < \varepsilon$. Then X has inverted inverse Gaussian distribution $g(x,\lambda_1,\mu_1)$ where $\lambda_1 = 4\mu/(2\rho)$ and $\mu_1 = \mu^{1/2}$.

Proof: By assumption $E(1/X)$ exists, let

$$\Psi(t) = \int \exp(itx)\ x^{-1}\ dG(x) \tag{3.22}$$

then $\Psi'(t) = i \int \exp(itx)\, dG(x) = ih(t)$ (3.23)
where $h(t)$ is the characteristic function of X.

From equations (3.21) and (3.23) we have that, in some neighbourhood of the origin

$$d/dt[\log\Psi(t)] = \Psi' / \Psi = i/(\mu - i\rho t)^{1/2} \tag{3.24}$$

$$\log\Psi(t) = k - (2/\rho)\ (\mu - i\rho t)^{1/2}$$

or $$\Psi(t) = \exp[k - (2/\rho)(\mu - i\rho t)^{1/2}] \tag{3.25}$$

Writing $k = 2\mu^{1/2}/\rho + \log\mu^{1/2}$ and making use of (3.21), we have

$$h(t) = \exp[2\mu^{1/2}/\rho + \log\mu^{1/2} - (2\mu^{1/2}/\rho)(1-i\rho t/\mu)^{1/2}] \,/[\mu^{1/2}(1-i\rho t/\mu)^{1/2}]$$

$$h(t) = (\exp[A(\mu,\rho)\mu^{-1/2} - A(\mu,\rho)\mu^{-1/2}((1-2it/\ A(\mu,\rho))^{1/2}])$$
$$/[1-2it/A(\mu,\rho)],$$

where $A(\mu,\rho) = 2\mu/\rho$.

Let $\lambda_1 = A(\mu,\rho)$ and $\mu_1 = \mu^{1/2}$, then we have

$$h(t) = \exp[\lambda_1/\mu_1 - \lambda_1/\mu_1(1-2it\lambda^{-1})^{1/2}]\ (1-2it/\lambda_1)^{-1/2}. \tag{3.26}$$

Equation (3.26) holds for all real $|t| < \varepsilon$ and $\varepsilon > 0$. By uniqueness theorem for characteristic functions it follows that the random variable X has inverted inverse Gaussian distribution. Since by assumption $E(1/x)$ exists, so $\Psi(t)$ is bounded.

3.6 Most Powerful Test

Let X be a random variable with density function

$$g(x,1,\mu) = (2\pi)^{-1}\ x^{-1/2} \exp[-(2x)^{-1}(x - 1/\mu)^2] \tag{3.27}$$

These densities have a monotone likelihood ratio in X. We show this directly by definition. Consider the ratio

$$R=L(\mu_1,x_1,\ldots,x_n)/L(\mu_2,x_1,\ldots,x_n)=\prod_{i=1}^{n} g(x_i,1,\mu_1)\ \Big/ \prod_{i=1}^{n} g(x_i,1,\mu_2)$$

Let $\mu_1 < \mu_2$ and $\mu_1, \mu_2 > 0$

Substituting the value of $g(x;1,\mu)$ and after some simplifications we obtain

$$R = \exp[-0.5(1/\mu_1^2 - 1/\mu_2^2) \sum_{i=1}^{n} (1/x_i) + n(1/\mu_1 - 1/\mu_2]$$

The ratio $L(\mu_1, x_1, \ldots, x_n)/L(\mu_2, x_1, \ldots, x_n)$ is a non-increasing function of the statistic $\sum_{i=1}^{n} (1/x_i)$ for every $\mu_2 > \mu_1 > 0$. Hence $g(x,1,\mu)$, for $x,\mu > 0$ has a monotone likelihood ratio in $\sum_{i=1}^{n} (1/x_i)$.

Therefore, if k is such that

$$P_{H_0}[\sum_{i=1}^{n} (1/x_i) > k] = \alpha$$

then the test corresponding to the critical region

$$C = [x_1, \ldots, x_n: \sum_{i=1}^{n} (1/x_i) > k]$$ is a uniformly most powerful test [38,p. 424] of size α of

$$H_0: \mu_1 \leq \mu_2 \quad \text{versus} \quad H_1: \mu_1 > \mu_2$$

3.7 Testing of Simple Hypotheses

Let $X_1, \ldots, X_n$ be independent random variables with common density function

$$g(x,1,\mu) = (\pi)^{-1} x^{-1/2} \exp[-(2x)^{-1}(x-1/\mu)^2] \text{ for } x,\mu > 0$$

Let the null hypothesis be

$$H_0: \mu = \mu_0$$

and the alternative hypothesis be

$$H_1: \mu \neq \mu_0 \quad \text{and} \quad \mu_0 > 0$$

Let α be a level of testing H_0 against H_1. We consider the test criterion to the above hypothesis by the fundamental lemma of Neyman [38,p.411].

$$L = \prod_{i=1}^{n} g(x_i,1,\mu_1)/ \prod_{i=1}^{n} g(x_i,1,\mu_2) > C$$

$$L=\exp[-(1/2) \sum_{i=1}^{n} (1/x_i)(x_i-1/\mu_1)^2 + (1/2) \sum_{i=1}^{n} (1/x_i) \quad (x_i-1/\mu_0)^2] > C \tag{3.28}$$

Taking log of both sides and after some simplifications we get

$$\sum_{i=1}^{n} (1/x_i) > C_1$$

$$H_n = n/ [\sum_{i=1}^{n} (1/x_i)] < K$$

where $K = n(1/\mu_0^2 - 1/\mu_1^2)/[2\log c + 2n(1/\mu_0 - 1/\mu_1)]$ and H_n is the harmonic mean of the n sample values. Thus we reject H_0 when $H_n < K$ and accept H_0 when $H_n \geq K$. Since H_n is also an inverted inverse Gaussian random variable with density function

$$g(H_n,n^2,n\mu) = [n^2/(2\pi)]^{1/2} (nH_n)^{-1/2} \exp[-(n/(2H_n))(H_n-\mu^{-1})^2]$$

where $H_n > 0$ and $\mu > 0$

we can evaluate K from the following integral equation

$$\int_K^{\infty} g(H_n,n^2,n\mu_0)\, dh_n = 1-\alpha \tag{3.29}$$

and $$\beta(\mu) = \int_K^{\infty} g(H_n,n^2,n\mu) \tag{3.30}$$

is called the power of the test and K is determined from equation (3.29).

4. INVERTED WEIBULL DISTRIBUTION

The Weibull is a prominent probability distribution in applied sciences, mainly in the analysis of extreme value phenomena and in reliability engineering. The Sweedish physicist Weibull popularized this model in 1939 when he introduced it on empirical grounds, in his analysis of material strength [39,40]. In this chapter we derive and discuss important properties of the inverted Weibull probability distribution which does not belong to the inverted Pearsonian class.

If Y is a two parameter Weibull random variable then its probability density function is

$$f(y,a,b) = ab\, y^{b-1} \exp(-ay^{b}) \qquad \text{for } y > 0 \tag{4.1}$$

The parameters a and b are the scale and shape parameters respectively. The graph of the density function in equation (4.1) is shown in Figure (4.1).

Using the reciprocal transformation $Y = 1/X$ for $X > 0$ the probability density function of inverted Weibull distribution is

$$g(x,a,b) = ab\, x^{-b-1} \exp(-ax^{-b}) \qquad \text{for } x > 0 \tag{4.2}$$

The distribution function of X is

$$G(x) = \exp(-ax^{-b}) \qquad x > 0 \tag{4.3}$$

4.1 Properties:

(i) The inverted Weibull distribution is unimodal and the mode is given by

$$X_{mode} = [ab/(b+1)]^{1/b} \qquad a,b > 0 \tag{4.4}$$

(ii) The median of the inverted Weibull distribution is given by the equation

$$\exp(-ax_m^{-b}) = 1/2 \tag{4.5}$$

Which has a solution $X_{med} = (a/0.30103)^{1/b}$

(iii) The graph of the probability density in (4.2) is drawn in Figure (4.2) for different values of the shape parameter b and a = 1.

(iv) The moments of the inverted Weibull random variable X are [41]:

$\mu_r' = E(X^r)$ where r is a positive integer.

$$= {}_0\int^\infty g(x)x^r\, dx$$

$$= ab\ {}_0\int^\infty x^{r-b-1}\exp(-ax^{-b})\, dx$$

let $y = a/x^b, \qquad 0 < y < \infty$

The transformation is one to one and the jacobian of the transformation is

$$dx/dy = -1/b\ (a)^{1/b}\ y^{-1/b-1}$$

therefore, $\mu'_r = {}_0\int^\infty a^{r/b}\ y^{-r/b}\ \exp(-y)\ dy$

$$= a^{r/b}\ \Gamma(1-r/b) \qquad r < b$$

The first two moments are

$$\mu'_1 = a^{1/b}\Gamma(1 - 1/b) \qquad \text{for } b > 1 \tag{4.6}$$

$$\mu'_2 = a^{2/b}\Gamma(1 - 2/b) \qquad \text{for } b > 2$$

The variance of the distribution is

$$\sigma^2(x) = a^{2/b}[\Gamma(1-2/b) - \Gamma^2(1-1/b)] \quad \text{for } b > 2 \tag{4.7}$$

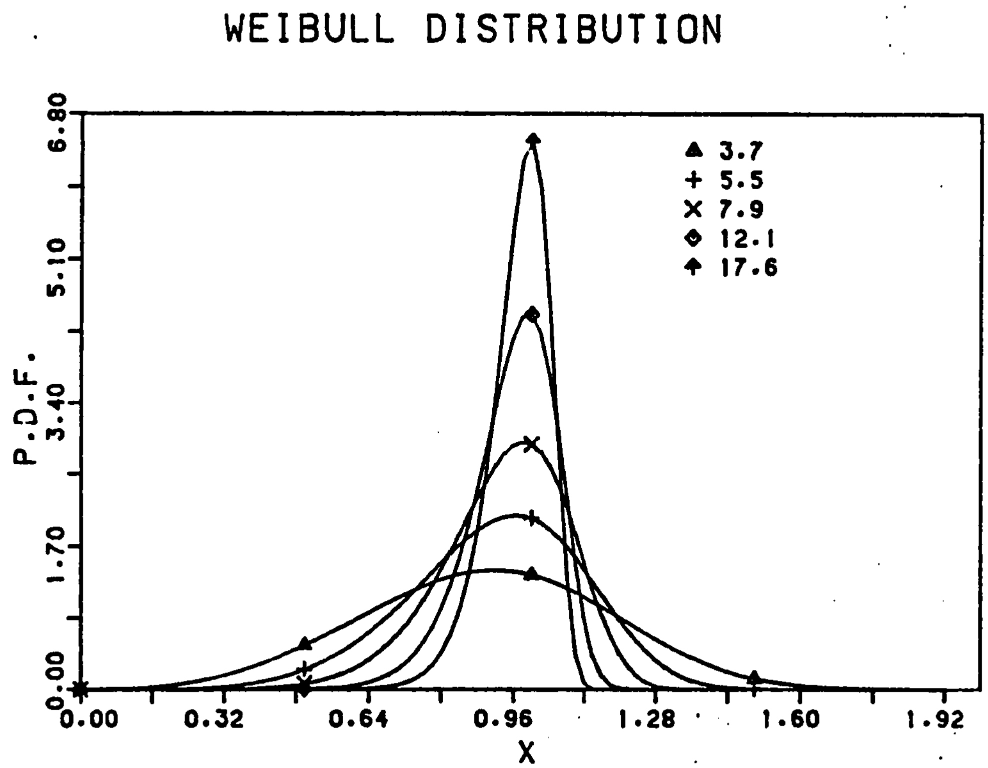

Fig. 4.1 The Weibull probability density function.

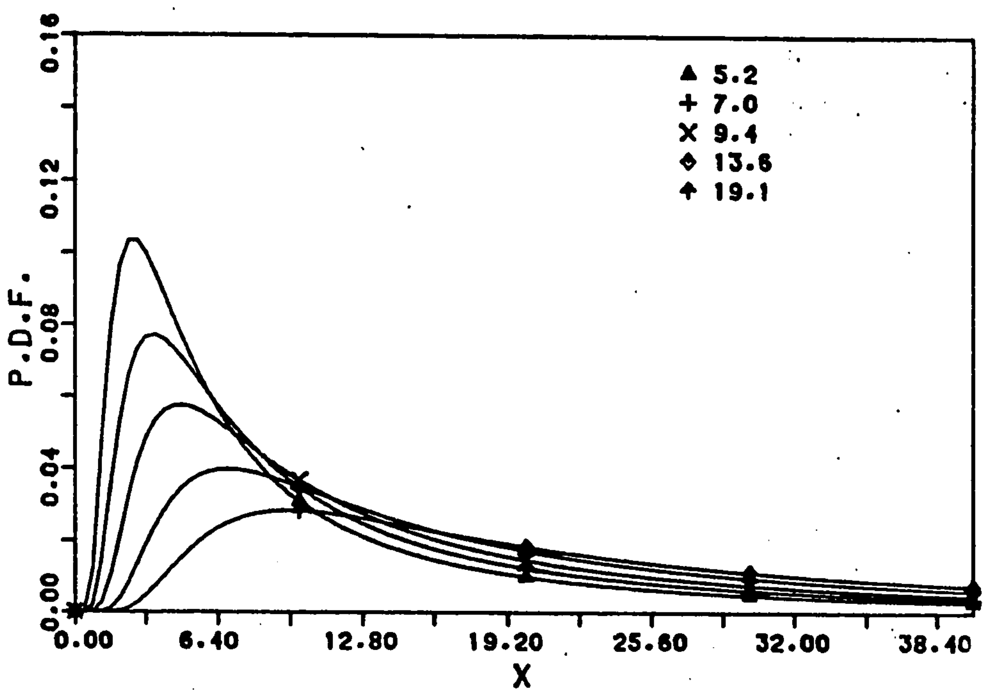

Fig. 4.2 The inverted Weibull probability density function.

(v) The coefficient of variation , γ , of the inverted Weibull random variable X is

$$\gamma = [\Gamma(1 - 2/b)/\Gamma^2(1 - 1/b) - 1]^{\frac{1}{2}} \qquad \text{for } b > 2 \qquad (4.8)$$

It is seen from equations (4.6) and (4.7) that the mean and variance of the distribution exist if $b > 1$ and $b > 2$ respectively. When b is very close to 2, the coefficient of variation is very high. However, the coefficient of variation is a decreasing function of the shape parameter $b > 2$. When $b \to +\infty$, then $\gamma \to 0$.

(vi) Sufficient Statistic

Like the Weibull model, no reduced sufficient statistics exist for this model.

(vii) Lower Bound For The Variance Of An Unbiased Estimator Of 'a':

Theorem:

For a random sample of size n from the inverted Weibull population with known shape parameter b, the statistic $T = (1/n)\sum_{i=1}^{n} (X_i)^{-b}$ is a uniformly minimum variance unbiased estimator of 1/a.

Proof:

Suppose a random sample of size n is drawn from an inverted Weibull population having probability density function as in (4.3).

let $\tau(a) = 1/a$ so that $\tau'(a) = -(1/a^2)$

The regularity conditions are satisfied and from (4.3)

$$\log g(x;a,b) = \log a + \log b - (b + 1)\log x - ax^{-b}.$$

$$\partial/\partial a[\log g(x,a,b)] = 1/a - x^{-b}$$

and $\partial^2 \log g(x;a,b)/\partial a^2 = -1/a^2$ gives

$$E_a[[\partial/\partial a(\log g)]^2] = \mathrm{Var}[X^{-b}] = 1/a^2$$

By theorm 3.3.2 , Cramer Rao lower bound for the variance of an unbiased estimator of 'a' is given by

$$\mathrm{Var}_a[T] \geq 1/(na^2)$$

$$\sum_{i=1}^{n} (\partial/\partial a)\log g = \sum_{i=1}^{n} (1/a - 1/x_i^{\,b})$$

$$= -n[(1/n)\sum_{i=1}^{n} x_i^{-b} - 1/a\ ]$$

For $K(a,n) = -n$, the Cramer Rao theorm implies that $(1/n)\sum_{i=1}^{n} x_i^{-b}$ is a uniformly minimum variance unbiased estimator of $1/a$.

4.2 Estimation Of Parameters

(a) Maximum likelihood Estimates of the Parameters a and b:

Suppose that a random sample of size n is drawn from the inverted Weibull population (4.2). The log of the likelihood function is

$$\log L(a,b) = n\log a + n\log b - (b+1)\sum_{i=1}^{n} \log x_i - a\sum_{i=1}^{n} (x_i)^{-b}.$$

On equating $\partial \log L(a,b)/\partial a$ and $\partial \log L(a,b)/\partial b$ to zero, noting that the second derivatives are negative and solving for a and b respectively, we obtain maximum likelihood estimating equations of a and b:

$$\hat{a} = n/[\sum_{i=1}^{n} (x_i)^{-\hat{b}}] \tag{4.9}$$

and
$$n/\hat{b} = \sum_{i=1}^{n} \log x_i - \hat{a}\sum_{i=1}^{n} [x_i^{-\hat{b}} \log x_i] \tag{4.10}$$

We show that the likelihood equation (4.10) has only one positive solution for b. Let

$y_i = \log x_i$

then equation (4.10) can be written as

$$(1/\hat{b}) - (1/n)\sum_{i=1}^{n} y_i + [\sum_{i=1}^{n} y_i\exp(-\hat{b}y_i)]/[\sum_{i=1}^{n} \exp(-\hat{b}y_i)] = 0 \tag{4.11}$$

equation (4.11) can be considered as a function $f(\hat{b})$ of $\hat{b}$.

$$f(\hat{b}) = (1/\hat{b}) - \bar{y} + [\sum_{i=1}^{n} y_i\exp(-\hat{b}y_i)]/[\sum_{i=1}^{n} \exp(-\hat{b}y_i)] \tag{4.12}$$

where $\bar{y} = (1/n)\sum_{i=1}^{n} y_i$

We notice that the roots of the equation (4.10) are the roots of $f(\hat{b})$. Substituting $w_i = y_i - \bar{y}$, $f(\hat{b})$ becomes

$$f(\hat{b}) = (1/\hat{b})-\bar{y}+[\sum_{i=1}^{n} (w_i+\bar{y})\exp[-\hat{b}(w_i+\bar{y})]]/[\sum_{i=1}^{n} \exp[-\hat{b}(w_i+\bar{y})]]=0$$

which can be simplified to

$$f(\hat{b}) = \sum_{i=1}^{n} \exp(-\hat{b}w_i)(1 + \hat{b}w_i) \tag{4.13}$$

It can be seen that

$\lim_{\hat{b}\to-\infty} f(\hat{b}) = -\infty$

and $\quad f'(\hat{b}) = -\hat{b}\sum_{i=1}^{n} w_i^2\exp(-\hat{b}w_i)$

$f'(0) = 0$ so the function $f(\hat{b})$ has a horizontal tangent at $\hat{b} = 0$. But

$f(0) = n \neq 0$

which implies that $\hat{b} = 0$ can not be a solution of the likelihood equation (4.10).

But $\quad f'(\hat{b}) < 0 \qquad$ for $\quad 0 < \hat{b} < +\infty$

and $\quad f'(\hat{b}) > 0 \qquad$ for $\quad -\infty < \hat{b} < 0$

Therefore, the function $f(\hat{b})$ has a maximum (=n) at $\hat{b}$ and a single negative and a single positive root.

(b) Method of moments estimates of a and b:

By definition, we have the equations

$$\tilde{a}^{1/\tilde{b}}\, \Gamma(1-1/\tilde{b}) = (1/n) \sum_{i=1}^{n} x_i \text{ for } \tilde{b} > 1 \tag{4.14}$$

and
$$\tilde{a}^{2/\tilde{b}}\, \Gamma(1-2/\tilde{b}) = (1/n) \sum_{i=1}^{n} x^2_i \text{ for } \tilde{b} > 2 \tag{4.15}$$

which give the moments estimates of the parameters a and b.

It is seen from equations (4.9) and (4.10) which give estimates of the parameters that these estimates are not in close form.

4.3 Distribution Of a given b

Consider $\quad Z_i = (X_i)^{-b} \qquad z_i > 0$

where the density of X_i is as in (4.2). The jacobian of the transformation is

$dx/dz = -1/b\ z_i^{-1/b-1}$

Therefore, the probability density of Z_i is

$f(z_i) = a \exp(-az_i) \qquad$ where $\quad z_i > 0$

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Let $Z = \sum_{i=1}^{n} Z_i$ where $Z_i = (X_i)^{-b}$

The moment generating function of Z will be

$$M_z(t) = [a/(a - t)]^n \quad \text{where} \quad t < a \tag{4.16}$$

which is the moment generating function of the gamma distribution with parameters n and a. Hence the distribution of Z is

$$h(z) = [a^n/\Gamma(n)]\ z^{n-1} \exp(-az) \quad \text{for } z > 0$$

Thus we have the following result:

Theorem:

Let X_i be an inverted Weibull random variable. Then $Z = \sum_{i=1}^{n} X_i^{-b}$ is a gamma random variable with parameters n and a.

4.4 P(Y<X) When Both X And Y Are Inverted Weibull Random Variables:

Let X and Y be independent inverted Weibull random variates with probability density functions $g_1(x,a,b)$ and $g_2(y,c,d)$ respectively. The joint probability density function of X and Y is

$$g(x,y) = abcd\ x^{-b-1}\ y^{-d-1} \exp[-(ax^{-b} + cy^{-d}] \tag{4.17}$$

let $V = X - Y$ then V varies from $-\infty$ to $+\infty$.

Substituting $x = v + y$ in equation (4.17), we obtain

$$g(v,y) = abcd\ (v + y)^{-b-1}\ y^{-d-1} \exp[-a(v + y)^{-b} - cy^{-d}]$$

$$\text{where} \quad -\infty < v < \infty \quad \text{and} \quad y > 0$$

$$P(Y < X) = P(V > 0) = \int_0^\infty \int_0^\infty g(v,y)\ dv\ dy$$

$$= abcd \int_0^\infty y^{-d-1} \exp(-cy^{-d})[\int_0^\infty (v + y)^{-b-1}$$

$$\exp[-a(v + y)^{-b} dv]]dy \qquad (4.18)$$

let $\quad F = \int_0^\infty (v + y)^{-b-1} \exp[-a(v + y)^{-b}]\, dv$

$$= \int_0^\infty 1/(ab)\ d/dv[\exp(-a(v + y)^{-b})]$$

$$= 1/(ab)\ \exp[-a(v + y)^{-b}]|_0^\infty = 1/(ab)\ [1-\exp(-ay^{-b})]$$

Substituting the value of F in (4.18) we obtain

$$P(Y < X) = cd \int_0^\infty y^{-d-1} \exp(-cy^{-d})\, dy - cd \int_0^\infty y^{-d-1}$$

$$\exp(-cy^{-d} - ay^{-b})\, dy$$

$$= cd(I_1 - I_2) \qquad (4.19)$$

Where $I_1 = \int_0^\infty y^{-d-1} \exp(-cy^{-d})\, dy$

$$= \int_0^\infty (1/c)\ d/dy(\exp(-cy^{-d})$$

$$= 1/(cd)$$

and $\quad I_2 = \int_0^\infty y^{-d-1} \exp(-cy^{-d} - ay^{-b})\, dy$

To solve this integral, we let b = d then

$$I_2 = \int_0^\infty y^{-d-1} \exp[-(c + a)y^{-d}]\, dy$$

$$= 1/[d(c + a] \int_0^\infty d[\exp[-(c+a)y^{-d})] = 1/[d(c + a)]$$

Substituting for I_1 and I_2 in (4.15), we get

$$P(Y < X) = a/(c + a)$$

4.5 Point Estimator and Confidence Limits For the Shape Parameter b

We define a random variable

$$R(n) = \hat{b}/b$$

where $\hat{b}$ = maximum likelihood estimate of b

We first prove that the distribution of R(n) is independent of the inverted Weibull population parameters b and X_p. Let

$$U_i = G(x_i) = \exp(-J(p)[x_i/X_p]^{-b}) \tag{4.20}$$

where $J(p) = \log(1/p)$

It is known [37,p.349] that U_i are random variables and the joint distribution of $U_1,\ldots,U_n$ is independent of the population parameters b and X_p. From (4.20) we have

$$\log u_i = -J(p)(x_i/X_p)^{-b}$$

or $$x_i = X_p[(1/J(p))\log(1/u_i)]^{-1/b} \tag{4.21}$$

Let $y_i = \log x_i$

then $$y_i = \log X_p - (1/b)Q_i \tag{4.22}$$

where $Q_i = \log[(1/J(p))\log(1/u_i)]$

Q_i is a function of u_i and u_i are independent of X_p and so Q_i are independent of b and X_p. From (4.22) we have

$$\bar{y} = \log X_p - (1/b)\bar{Q} \tag{4.23}$$

where $\bar{y} = (1/n)\sum_{i=1}^{n} y_i$ and $\bar{Q} = (1/n)\sum_{i=1}^{n} Q_i$

Let $Z_i = y_i - \bar{y}$

$$= -(1/b)(Q_i - \bar{Q}) \tag{4.24}$$

But from (4.13) we know that $\hat{b}$ is the solution of

$$\sum_{i=1}^{n} \exp(-\hat{b}z_i)(1 - \hat{b}z_i) = 0 \tag{4.25}$$

Substituting (4.24) in (4.25) we get

$$\sum_{i=1}^{n} \exp(Q_i - \overline{Q})R[1 - (Q_i - \overline{Q})R] = 0 \tag{4.26}$$

From (4.26) it is clear that R is an explicit function of the random variables $[Q_i]$. But Q_i,s have a joint distribution independent of X_p and b. So the distribution of R(n) will depend upon n only. This means that R(n) can be regarded as a test statistic and the distribution of R(n) can be found by monte carlo simulation. Let R_α be the αth percentile of the distribution of R(n). Then we write

$$Pr[R(n)= \hat{b}/b< R_\alpha] = \alpha$$

or $$Pr[b > \hat{b}/R_\alpha] = \alpha \tag{4.27}$$

Which gives the one sided 100(1 - α) percent lower confidence interval for b. Similarly one sided upper and two sided confidence limits for b can be found using the test statistics R(n).

4.6 Test of Hypothesis About the Shape Parameter b

The test statistic R(n) can be used to test a hypothesis concerning the inverted Weibull shape parameter b. Let the null hypothesis be

$$H_0: b = b_0$$

If we consider the alternate hypothesis as

$$H_1: b < b_0$$

at 100α percent level of significance. Then the acceptance region can be found as

$$Pr(\text{To accept } H_0 \text{ when } H_1 \text{ is true}) = 1 - Pr[R_\alpha(n) > \hat{b}/b_0]$$

Since H_1 is true, so we can write $b = kb_0$

$$= 1 - Pr[R_\alpha/k > \hat{b}/b]$$

$$= 1 - G_R[R_\alpha(n)/k] \qquad (4.28)$$

Therefore, in this case we will accept the null hypothesis if $R_\alpha(n) < (\hat{b}/b_0)$. The other two possibilities for the alternative hypothesis and the corresponding critical regions at 100α percent level of significance are given below.

Alternate Hypothesis	Critical Region
H_1: $b > b_0$	$\hat{b}/b_0 < R_{1-\alpha}(n)$
H_1: $b \neq b_0$	$R_{\alpha/2}(n) < \hat{b}/b_0 < R_{1-\alpha/2}(n)$

4.7 Inference About the Inverted Weibull Quantile

Let $\hat{X}_p$ be the maximum likelihood estimate of the pth order quantile of the inverted Weibull distribution. We define a random variable W as

$$W(n,p) = \hat{b}\,\log(\hat{X}_p/X_p) \qquad (4.29)$$

We first prove that w(n,p) is independent of the inverted weibull population parameters. From (4.9) and the maximum likelihood estimate for the pth quantile

$\hat{X}_p = (\hat{a})^{1/\hat{b}}\,[J(p)]^{-1/\hat{b}}$ we have

$$(\hat{X}_p)^{\hat{b}} = \hat{a}[J(p)]^{-1}$$

$$= [n/J(p)]\,[\sum_{i=1}^{n} (x_i)^{-\hat{b}}]^{-1}$$

$$= [n/J(p)]\,[\sum_{i=1}^{n} \exp(-y_i\hat{b})]^{-1}$$

$$= [n/J(p)]\,[\sum_{i=1}^{n} \exp(-y_i\hat{b}R)]^{-1} \quad \text{(because } y_i = \log x_i) \qquad (4.30)$$

substituting (4.22) in (4.30)

$$(\hat{X}_p)^{\hat{b}} = [n/J(p)]\ [\sum_{i=1}^{n} \exp(-\hat{b}R(\log\hat{X}_p - Q_i/b))]^{-1}$$

$$= [n/J(p)]\ [(1/\hat{X}_p)^{bR} \sum_{i=1}^{n} \exp(RQ_i)]^{-1}$$

$$= [n(\hat{X}_p)^{\hat{b}R}/J(p)]\ [\sum_{i=1}^{n} \exp(RQ_i)]^{-1}$$

$$(\hat{X}_p/X_p)^{\hat{b}} = [n/J(p)]\ [\sum_{i=1}^{n} \exp(RQ_i)]^{-1} \tag{4.31}$$

So W is given by

$$W(n,p)= \log[\hat{X}_p/X_p]^{\hat{b}}$$

$$= \log[(n/J(p)(1/\sum_{i=1}^{n} \exp(RQ_i))] \tag{4.32}$$

but R and Q_i are independent of the population parameters b and X_p. Hence W depends upon n and p only. The distribution of W(n,p) can thus be found for given n and p. Using the definition of W we can write the confidence limits upon X_p. The one sided 100(1-α) percent lower confidence limit is given as

$$\Pr[\hat{b}\ \log(\hat{X}_p/X_p) < w_{1-\alpha}(n,p)] = 1 - \alpha \tag{4.33}$$

or

$$\Pr[\hat{X}_p \exp(-w_{1-\alpha}/\hat{b}) < X_p] = 1 - \alpha$$

The distribution of the statistic W(n,p) can also be used to test hypothesis about the inverted weibull quantile. Suppose the null hypothesis is

$$H_0: X_p = X_{p_0}$$

$$H_0: X_p < X_{p_0}$$

The acceptance region for a 100(1 - α) percent significant test is given by

$$\Pr[\text{To accept } H_0 \text{ when } H_1 \text{ is true}] = \Pr[\hat{b}\log(\hat{X}_p/X_{p_0}) > w_\alpha] \qquad (4.34)$$

Similarly for the other alternative hypothesis the acceptance regions are given below.

Alternate Hypothesis H_1	H_0 is accepted if
$X_p > X_{p_0}$	$\hat{b}\log(\hat{X}_p/X_{p_0}) < w_{1-\alpha}(n,p)$
$X_p \neq X_{p_0}$	$w\alpha/2(n,p) < \hat{b}\log(\hat{X}p/Xp0) < w1\text{-}\alpha/2(n,p)$

5. *INVERTED GAMMA DISTRIBUTION*

We consider the following density function to represent the two parameter inverted gamma distribution with parameters λ and r.

$$g(x,\lambda,r) = \lambda^r/(\Gamma(r))\ x^{-r-1}\ \exp(-\lambda/x),\ \text{where } \lambda,\ r,\ x > 0 \qquad (5.1)$$

where λ is a scale parameter and and r is a shape parameter.

5.1 *Properties:*

(i) The inverted gamma distribution is unimodal and the unique mode lies at

$$X_{mode} = \lambda/(r+1)$$

(ii) The median of the distribution is not in close form and is given by the solution of the equation

$$\int_0^x y^{-r-1}\ \exp(-\lambda/y) = (1/2)\Gamma(r)/\lambda^r$$

(iii) Due to the usefulness of the negative moments in applied sciences, negative moments of the gamma distribution which are the same as the positive moments of the inverted gamma distribution are given in [41]. The moment generating function of the inverted gamma distribution is

$$E(X^n) = \lambda^n\ \Gamma(r - n)/\Gamma(r) \quad \text{for } n < r \qquad (5.2)$$

Thus $\quad E(X) = \lambda/(r-1) \quad \text{for } r > 1$

$$\text{and } Var(X) = \lambda^2/[(r-1)^2(r-2)] \quad \text{for } r > 2$$

(iv) The moments of the gamma distribution can can also be found using the Chao and Strawderman method [23]. The probability density

function of a two parameter gamma distribution is

$$f(y) = (\lambda^r/\Gamma(r))\ y^{r-1} \exp(-\lambda y) \qquad x, r, \lambda > 0$$

and $g_1(t) = E(t^{y+A-1}) \quad 0 < t \leq 1$

$$= t^{A-1}\ \lambda^r/\Gamma(r)\ {}_0\!\int^{\infty} y^{r-1} \exp[-(\lambda - \ln t)y]\ dy$$

Let $X = (\lambda - \ln t)Y$. We have $dx = (\lambda - \ln t)dy$.

So $g_1(t) = t^{A-1}\ \lambda^r/(\lambda - \ln t)^r$.

For $A = 0$, $g_1(t) = \lambda^r\ t^{-1}/(\lambda - \ln t)^r$.

Also, $g_2(t) = 1/t\ {}_0\!\int^t g_1(u)\ du$

$$= \lambda^r(\lambda - \ln t)^{1-r}/[t(r-1)] \quad \text{for } r > 1$$

Also, $E(1/y) = {}_0\!\int^1 \lambda^r(\lambda - \ln t)^{-r}\ 1/t\ dt$

$$= \lambda^r[(\lambda - \ln t)^{-r+1}\ (r-1)^{-1}]_0^1 \quad \text{for } r > 1$$

$$= \lambda/(r-1) \quad \text{for } r > 1.$$

Similarly,

$E[(1/y)^2] = {}_0\!\int^1 g_2(t)\ dt$

$$= \lambda^2[(r-1)(r-2)]^{-1} \quad \text{for } r > 2$$

The higher moments can be calculated by integrating $g_k(u)$ for $k = 3, 4, \ldots$.

(*v*) The coefficient of variation of the distribution is $(r - 2)^{-1/2}$.

(*vi*) The equations giving the maximum likelihood estimates of the parameters λ and r are

$$\hat{r} = (\hat{\lambda}/n) \sum_{i=1}^{n} (x_i)^{-1} \tag{5.3}$$

$$\hat{\lambda} = \exp[\Gamma'(\hat{r})/\Gamma(\hat{r})] + (1/n) \sum_{i=1}^{n} \log x_i] \tag{5.4}$$

In this case, the likelihood equation (5.4) is not easily solvable and it is necessary to resort to numerical methods, using table for $\Gamma'(r)/\Gamma(r)$.

(*vii*) The method of moments estimates of the parameters r and λ can be found more easily. For a sample of size n from an inverted gamma population, moment estimating equations are

$$(1/n) \sum_{i=1}^{n} x_i = \tilde{\lambda}/(\tilde{r}-1) \tag{5.5}$$

$$(1/n) \sum_{i=1}^{n} x^2_i = \tilde{\lambda}^2/[(\tilde{r}-1)(\tilde{r}-2)] \tag{5.6}$$

Solving equations (5.5) and (5.6) simultanously, we obtain

$$\tilde{r} = \overline{X}^2/S^2 + 2 \tag{5.7}$$

$$\tilde{\lambda} = \overline{X}(\tilde{r}-1) \tag{5.8}$$

Equations (5.7) and (5.8) give the moments estimates of the parameters. These equations are simple and are in explicit form.

(*viii*) ***Theorem:*** If X is an inverted gamma random variable with probability density function $g(x,r,\lambda)$ then

$$Y = aX + b \quad a > 0,\ b \geq 0$$

is an inverted gamma random variate with probability density function $g(x - b, r, \lambda a)$.

Proof: The probability density function of X is

$$g(x,r,\lambda) = [\lambda^r/\Gamma(r)]\ x^{-r-1} \exp(-\lambda/x) \text{ where } x,r,\lambda > 0 \tag{5.9}$$

Let $\quad Y = aX + b$

$$\text{when } x \to 0, \text{ then } y \to b$$

$$\text{when } x \to +\infty, \text{ then } y \to +\infty$$

$$x = (y - b)/a \quad \text{and} \quad dx/dy = 1/a, \quad a > 0$$

The probability density function of Y is

$$g(y) = \lambda^r/[\Gamma(r)]\ a_r(y - b)^{-r-1} \exp[-\lambda a/(y - b)] \text{ where } a < y < +\infty.$$

(ix) Consider $T(x_1,..,x_n) = (1/n) \sum_{i=1}^{n} \ln x_i$ and $\tau(r) = \Gamma'(r)/\Gamma(r)$.

so that $\quad \tau'(r) = d/dr[\Gamma'(r)/\Gamma(r)]$

$E[T(x_1,..,x_n)] = \tau(r)$ so $T(x_1,..,x_n)$ is an unbiased estimator of $\tau(r)$.

Let $X_1,...,X_n$ be a random sample from an inverted gamma population having probability density as in (5.1). Then

$$\partial/\partial r[\log g] = \log(\lambda/x) - d/dr[\log\Gamma(r)]$$

$$E[\partial/\partial r \log g]^2 = E[\log(\lambda/X) - d/dr[\log\Gamma(r)]^2$$

$$= \mathrm{Var}[\log(\lambda/X)]$$

Let $W = \log(\lambda/X)$. The probability density of W is given by

$$h(w) = [\Gamma(r)]^{-1} \exp[rw - \exp(w)] \quad -\infty < w < +\infty \tag{5.10}$$

which is the probability density of the extreme value random variable.

The moment generating function of the random variable Z can be found as follows.

$$M(t) = E(Z) = E[\exp(tW)]$$

$$= [\Gamma(r)]^{-1} \int_{-\infty}^{+\infty} \exp(tw)\ \exp[rw - \exp(w)]dw \qquad (5.11)$$

Let $W = \log Y$ or $Y = \exp(W)$ $y > 0$

$dw = (1/y)dy$

substituting values in equation (5.11) and after some simplifications we obtain

$$M(t) = \Gamma(r+t)/\Gamma(r) \qquad (5.12)$$

Using equation (4.52), the mean and the variance of W turns out to be

$E[W] = d/dr[\log\Gamma(r)]$

and $\mathrm{Var}(W) = [\Gamma(r)]^{-1}\ d^2/dr^2[\Gamma(r)] - [d/dr \log\Gamma(r)]^2$

Hence the Cramer-Rao lower bound for the variance of an unbiased estimator T of the shape parameter r is given by

$$\mathrm{Var}_r[T] = \tau'/[n(\Gamma(r))^{-1}\ d^2/dr^2\Gamma(r) - (d/dr\Gamma(r))^2]]$$

From equation (5.10), we notice the following relationship between the inverted gamma and the extreme value distribution.

Theorem: Let X be an inverted gamma random variable. If

$$W = \log(\lambda/X)$$

then W is an extreme value random variable with probability density

$$h(w) = [1/\Gamma(r)]\ \exp[rw-\exp(w)] \qquad -\infty < w < +\infty$$

5.2 Inverted Chi-Square Distribution

The probability density function of the inverted gamma distribution is

$$g(x) = [\lambda^r/\Gamma(r)]\ x^{-1-r} \exp(-\lambda/x) \qquad \lambda, r, x > 0$$

substituting $r = k/2$ and $\lambda = 1/2$ we obtain

$$g(x) = [1/\Gamma(k/2)\ 2^{-k/2}\ x^{-1-k/2} \exp[-1/(2x)] \text{ for } k, x > 0 \qquad (5.13)$$

The probability density in (5.13) is the inverted chi-square density function.

In this section we show that this probability density can be used to find the interval estimate for the variance of a normal population with known mean.

Let $X_1,\dots,X_n$ be a random sample from a normal population $N(\mu,\sigma^2)$ where μ is known. Let X and S^2 denote the mean and the variance of the sample. The statistic $\sigma^2/(nS^2)$ has an inverted chi-square $(i\chi^2)$ distribution with $(n-1)$ degrees of freedom. To construct the confidence limits for σ^2, let p and q be such that

$$_0\int^p g(i\chi^2)\ d(I\chi^2) = \alpha_1$$

and

$$_q\int^\infty g(i\chi^2)\ d(i\chi^2) = \alpha_2$$

where $\alpha = \alpha_1 + \alpha_2$. We then have $(1-\alpha)$% percent confidence interval for $\sigma^2/(nS^2)$ as

$$P[p < \sigma^2/(nS^2) < q] = 1-\alpha$$

$$\text{or} \quad P[nS^2p < \sigma^2 < nS^2q] = 1-\alpha$$

The values of p and q are obtained from the Table (5.1) of the inverted chi-square distribution($i\chi^2$) with $(n-1)$ degrees of freedom which are found as

$$p = i\chi^2_{\alpha_1} \quad \text{and} \quad q = i\chi^2_{1-\alpha_2}$$

Hence $(1-\alpha)\%$ confidence interval for population variance σ^2 of a normal population is given by

$$i\chi^2_{\alpha_1} \sum_{i=1}^{n} (x_i - X)^2 < \sigma^2 < \sum_{i=1}^{n} (x_i - X)^2 \; i\chi^2_{1-\alpha_2}$$

5.3 Characterizations Of the Inverted Gamma Distribution

The inverted gamma distribution is the limiting distribution in μ for the inverse Gaussian distribution. The following result shows that for large values of the mean μ of the inverse Gaussian distribution, inverted gamma distribution is the natural approximation to the inverse Gaussian distribution.

Theorem 5.3.1: Let X be an inverse Gaussian random variable with probability density $g(x,\lambda,\mu)$ and Y be an inverted gamma random variable with probability density $f(y,\lambda)$. Then as $\mu \to +\infty$ the distribution of X approaches the distribution of Y.i.e.

$$\lim_{\mu \to +\infty} g(x,\lambda,\mu) = f(y,\lambda)$$

Proof: The probability density function $g(x,\lambda,\mu)$ of X is

$$g(x,\lambda,\mu) = [\lambda/(2\pi)]^{1/2} \, x^{-3/2} \exp[-\lambda/(2x)] \, h(x,\lambda,\mu)$$

$$\text{where } h(x,\lambda,\mu) = \exp(-\lambda/(2\mu^2)x + \lambda/\mu)$$

Let $\Lambda = -\lambda x/(2\mu^2) + (\lambda/\mu)$

TABLE 5.1 The Inverted Chi-Square Distribution

$$Pr(X\leq x)= {}_0\int^x f(i\chi^2)\, d(\chi^2)$$

n	.010	.025	.050	.100	.250	.5000	.950	.975	.990
1	.150	.199	.260	.369	.755	2.197	8.000	8.000	8.000
2	.108	.135	.166	.217	.360	0.721	8.000	8.000	8.000
3	.088	.106	.127	.159	.243	0.422	2.841	4.633	8.000
4	.075	.089	.105	.128	.185	0.297	1.407	2.064	3.366
5	.066	.077	.090	.108	.150	0.229	0.873	1.203	1.804
6	.059	.069	.079	.093	.127	0.186	0.611	0.808	1.146
7	.054	.062	.071	.083	.110	0.157	0.461	0.591	0.807
8	.049	.057	.064	.074	.097	0.136	0.365	0.458	0.607
9	.046	.052	.059	.068	.087	0.119	0.300	0.370	0.479
10	.043	.048	.054	.062	.079	0.107	0.253	0.307	0.390
11	.040	.045	.050	.057	.072	0.096	0.218	0.262	0.327
12	.038	.042	.047	.053	.067	0.088	0.191	0.227	0.280
13	.036	.040	.044	.050	.062	0.081	0.169	0.199	0.243
14	.034	.038	.042	.047	.058	0.074	0.152	0.177	0.214
15	.032	.036	.040	.044	.054	0.069	0.137	0.159	0.191
16	.031	.034	.038	.042	.051	0.065	0.125	0.144	0.172
17	.029	.033	.036	.040	.048	0.061	0.115	0.132	0.156
18	.028	.031	.034	.038	.046	0.057	0.106	0.121	0.142
19	.027	.030	.033	.036	.044	0.054	0.098	0.112	0.131
20	.026	.029	.031	.035	.041	0.051	0.092	0.104	0.121
21	.025	.028	.030	.033	.040	0.049	0.086	0.097	0.112
22	.024	.027	.029	.032	.038	0.046	0.081	0.091	0.104
23	.024	.026	.028	.031	.036	0.044	0.076	0.085	0.098
24	.023	.025	.027	.030	.035	0.042	0.072	0.080	0.092
25	.022	.024	.026	.029	.034	0.041	0.068	0.076	0.086
26	.021	.023	.025	.028	.032	0.039	0.065	0.072	0.081
27	.021	.023	.024	.027	.031	0.037	0.061	0.068	0.077
28	.020	.022	.024	.026	.030	0.036	0.059	0.065	0.073
29	.020	.021	.023	.025	.029	0.035	0.056	0.062	0.070
30	.019	.021	.022	.024	.028	0.034	0.054	0.059	0.066

then $\lim_{\mu\to+\infty} \Lambda = 0$

But $\exp(\Lambda)$ regarded as a function of μ is continous at zero. Then

$\lim_{\mu\to+\infty} \exp(\Lambda) = 1$

so $\lim_{\mu\to+\infty} g(x,\lambda,\mu) = f(y,\lambda)$

where $f(y,\lambda) = [\alpha^r/\Gamma(r)]\ y^{-r-1} \exp(-\alpha y^{-1})$
for $r = 1/2$ and $\alpha = \lambda/2$

Theorem 5.3.2: Let X be a positive random variable with distribution function F. Suppose that $E[\exp(X)] = \mu > 0$ exists and for some real $\eta \neq 0$, $\eta > 0$ and every real t the relation

$$it\eta \int \exp(itx)f(x)\,dx = \int \exp(itx)[\exp(x) - \mu]\, f(x)dx \qquad (5.14)$$

be satisfied. Then F is absolutely continuous and the distribution of Y = exp(-X) is inverted gamma.

Proof: Multiplying both sides of equation (5.14) by

$$[\exp(-its) - \exp(-ith-its)]/[2\pi(it)^2]$$

and integarting with respect to s and t over the intervals (α,β) and $(T,-T)$ respectively. The right hand side (R.H.S) is equal to

$$R.H.S = (2\pi)^{-1} \int_{-T}^{T} [1 - \exp(-iht)]/(it) \int_{\alpha}^{\beta} \exp(-it\ s)\, ds$$

$$\int_0^{\infty} \exp(itx)[\exp(x) - \mu]\ f(x)\ dx \qquad (5.15)$$

$$= (2\pi)^{-1} \int_{\alpha}^{\beta} ds\Big(\int_{-T}^{T}[\exp(-its[1 - \exp(-iht)]/\ (it)$$

$$\int_0^{\infty} \exp(itx)[\exp(x) - \mu]f(x)\ dx]\Big) \qquad (5.16)$$

Now

$$|\exp(-it(s-x))[1 - \exp(-iht)/(it) = \Big|\int_{(s-x)}^{(s+h-x)} \exp(-ity)dy\Big|$$

$$\leq h+s-x-s+x = h$$

and

$$_{-T}\int^{T}[_{-\infty}\int^{+\infty} h\ f(x)\ dx]dt = 2Th[F(+\infty) - F(-\infty)] < \infty$$

By Fubini's theorem, the order of integration may be interchanged in (5.15) to obtain (5.16). Letting $T\to\infty$ in (5.16) we obtain

$$R.H.S = \lim_{T\to\infty}(2\pi)^{-1}\ _{\alpha}\int^{\beta} ds[_{-T}\int^{T}\exp(-its)[1-\exp(-iht)]/(it)$$

$$_{0}\int^{\infty} \exp(itx)[\exp(x) - \mu]\ dF(x)]\ dt$$

$$= \lim_{T\to\infty}\ (2\pi)^{-1}\ _{\alpha}\int^{\beta} ds\ \int([\exp(x) - \mu]f(x)d\ x$$

$$_{-T}\int^{T}\ [\exp(-its + itx) - \exp(-ith-ity+its)]/(it)\ dt \qquad (5.17)$$

$$R.H.S = {}_{\alpha}\int^{\beta}\ _{0}\int^{\infty} J_T(s,h)\ [\exp(x) - \mu)dF(x) \qquad (5.18)$$

where $J_T(s,h) = [\lim_{T\to\infty}(2\pi)^{-1}\ _{-T}\int^{T}\ [\exp(-its + itx)-$

$$\exp(-its + itx - ith)]/(it)dt]]dF(x)$$

$$= \lim_{T\to\infty}(2\pi)^{-1}\ _{-T}\int^{T}[\exp(x)-\mu][\sin(t(s - x)-$$

$$\sin t(s-x+h)]/t\ dt$$

because

$$_{-T}\int^{T}\ [\cos t(s - x)-\cos t(s + h - x)]/(iu)\ dt = 0$$

Because the integrand is an odd function.

Let $\nu = t(s - x)$ and $\omega = t(s - x + h)$

Then $J_T(s,h) = {}_{-\infty}\int^{\infty} [\exp(x) - \mu] \lim_{T\to\infty}(2\pi)^{-1}$

$[\ {}_{-T(s-x)}\int^{T(s+h-x)} \sin\nu/\nu \, d\nu - {}_{-T(s-x)}\int^{T(s+h-x)} \cos\omega/\ \omega \, d\omega] dF(x)$

But $\lim_{T\to\infty}\ {}_{T}\int^{T} \sin\nu/\nu \, d\nu \to \pi$

and $\lim_{T\to\infty}\ {}_{T}\int^{T} \sin\omega/\omega \, d\omega \to \pi$

and since the integral is continuous in T, it is bounded uniformly in T. Thus for some $M < \infty$

$|J_T(s,h)| \leq M$ for all T and X.

Therefore, $J_T(s,h) \to J$ when $T \to \infty$

where

$$J(x) = \begin{cases} 0 & \text{if } x < s + h \text{ or } x > s + h \\ 1 & \text{if } s < s < s + h \\ 1/2 & \text{if } x = s + h \text{ or } x = s \end{cases}$$

Therefore, by dominated convergence theorem

$$R.H.S = {}_{\alpha}\int^{\beta} [{}_{0}\int^{\infty} J(x)[\exp(x) - \mu] \, dF(x)] ds$$

$$= {}_{\alpha}\int^{\beta} f(s,h) \, ds \qquad (5.19)$$

where

$$f(s,h) = {}_{0}\int^{\infty} j(x)[\exp(x) - \mu] \, dF(x)$$

$$f(s,h) = F(s+h)^{-} - F(s) + 1/2[F(s) - F(s^{-}) + F(s+h) - F(s+h)^{-}]$$
$$= F(s+h) - F(s)$$

Substituting values in (5.19) , the right hand side becomes

$$R.H.S = {}_{\alpha}\int^{\beta}[{}_{s}\int^{(s+h)} [\exp(x)-\mu]\ dF(x)]ds \tag{5.20}$$

Where the left hand side (L.H.S) is

$$L.H.S = \lim_{T\to\infty} {}_{-T}\int^{T} {}_{\alpha}\int^{\beta} [\exp(-its) - \exp(-ith-its)\ ]\ [2\pi(it)^2]^{-1}$$

$$i\eta t\ h(t)\ ds\ dt$$

$$= \lim_{T\to\infty} (2\pi)^{-1} {}_{-T}\int^{T} {}_{\alpha}\int^{\beta}\exp(-its)[1-\exp(\ -it)]/(it)^{-1}$$

$$\eta\ h(t)\ ds\ dt$$

$$= \lim_{T\to\infty}\eta(2\pi it)^{-1} {}_{-T}\int^{T} [\exp(-i\alpha t)-\exp(-i\beta t)]$$

$$[1 - \exp(-iht)]\ h(t)dt$$

$$= \eta\lim_{T\to\infty}(2\pi it)^{-1} {}_{-T}\int^{T} [\exp(-i\alpha t)-\exp(-it\alpha-i\ th)$$

$$-\exp(-i\beta t) + \exp(-it\beta - ith)]\ h(t)\ dt$$

Using inversion formula for characterstic functions we obtain

$$L.H.S = \eta[F(\alpha + h) - F(\alpha)] - \eta[F(\beta + h) - F(\beta)] \tag{5.21}$$

Equating (5.20) and (5.21) and letting $h\to-\infty$

$${}_{\alpha}\int^{\beta}[]_{s}\int^{-\infty} (\exp(x) - \mu)\ dF(x)]ds = F(\beta) - F(\alpha)$$

$$= \eta\ {}_{\alpha}\int^{\beta} dF(x) \tag{5.22}$$

$$\text{or}\quad (1/n)\ {}_{\alpha}\int^{\beta} f(s)\ ds = F(\beta) - F(\alpha) \tag{5.23}$$

$$\text{where}\quad f(s) = -(1/\eta)\ {}_{-\infty}\int^{s} [\exp(x) - \mu)\ dF(x) \tag{5.24}$$

Equation (5.22) shows that F is absolutely continuous with distribution function F. Differentiating (5.24) with respect to s

$f'(x) = -(1/\eta)]\exp(x) - \mu]\ f(x)$

$= -(1/\eta)\ \exp[\exp(x) - \mu x - k)]$

Substituting $y = \exp(-x)$ for which the jacobian of the transformation is $-1/y$. The density function of Y is

$f(y) = k\ y^{-1-\mu}\ \exp(-\eta/y) \qquad \eta, \mu > 0.$

6. INVERTED BURR DISTRIBUTION

If a random variable is equal in distribution to its own reciprocal, we say that it has reciprocal property. Most of the well known probability models do not posses this property. Such a reciprocal law occur infrequently. Possessing the reciprocal property is equivalent with the logarithm of that random variable being symmetrical about zero. This result is given by Seshadri[42]. It is interesting to note that for $\beta = 1$ the inverted Burr distribution is the same as Burr distribution. Thus for $\beta = 1$ the reciprocal property is enjoyed by the inverted Burr probability model. The probability density function of inverted Burr distribution is

$$g(x) = \alpha\beta[(1 + x^{\alpha}]^{-1-\beta}\ x^{\alpha\beta-1} \text{ where } 0 < x < \infty\ ,\ \alpha,\beta > 0 \tag{6.1}$$

Burr [43] derived this distribution from a differential equation. Consider the differential equation

$$dF = F(1 - F)g(y)\ dy \tag{6.2}$$

where $g(y)$ is some convenient function, which must be non negative in $0 \le F \le 1$ and the range of y. The equation (6.2) may be written as

$$dF[1/F + 1/(1 - F)] = g(y)dy$$

which has a solution $F(y) = [1 + \exp(-G(y))]^{-1}$ (6.3)

where $G(y) = \int_{-\infty}^{y} g(t)\ dt$

One convenient form of the solution of (6.2) is

$$F(x) = 1 - (1+x^{\alpha})^{-\beta} \qquad \text{for} \quad 0 \le x \le \infty \quad \alpha,\beta > 0$$

Some other solutions of (6.2) are also admissible [43].

6.1 Properties:

(i) The moment generating function of the inverted Burr random variable X is

$$M_X(t) = E[\exp(tX)]$$

$$= \alpha\beta \int_0^{\infty} \exp(tx)\ x^{\alpha\beta-1}\ (1 + x^{\alpha})^{-1-\beta}\ dx$$

$$= \alpha\beta \sum_{j=0}^{\infty} (t^n/n!) \int_0^{\infty} (1 + x^{\alpha})^{-1-\beta}\ x^{n+\alpha\beta-1} dx$$

Let $y = (1+x^{\alpha})^{-1}$ $\quad 0 \le y \le 1$ then $x = (1-y)^{1/\alpha}\ y^{-1/\alpha}$,

and $dx/dy = (-1/\alpha)\ (1-y)^{1/\alpha-1}\ y^{-1/\alpha-1}$

Therefore, $M(t) = \beta \sum_{j=0}^{\infty} (t^n/n!) \int_0^1 (1 - y)^{\beta+n/\alpha-1}\ y^{-n/\alpha} dy$

$$= \beta \sum_{j=0}^{\infty} (t^n/n!)\ B(1-n/\alpha\ ,\ \beta+n/\alpha) \quad \text{for } \alpha > n$$

$$= \beta \sum_{j=0}^{\infty} (t^n/n!)\ [\Gamma(1-n/\alpha).\ \Gamma(\beta+n/\alpha)]/\Gamma(1+\beta) \quad \text{for } \alpha > n$$

The moments of the inverted Burr distribution can also be calculated by using the definition.

$$\mu'_r = E[X^r] = \int_0^{\infty} \alpha\beta\ x^r\ (1+x^{\alpha})^{-1-\beta}\ x^{\alpha\beta-1}\ dx$$

$$= \alpha\beta \int_0^{\infty} (1 + x^{\alpha})^{-1-\beta}\ x^{\alpha\beta+r-1}\ dx \tag{6.4}$$

Let $y = (1 + x^{\alpha})^{-1}$ $\quad 0 \le y \le 1$

$$x = (1 - y)^{1/\alpha}\ y^{-1/\alpha}$$

$$dx/dy = -1/\alpha\ (1-y)^{1/\alpha-1}\ y^{-1/\alpha-1}$$

Substituting values in (6.4) and after some simplifications we get

$$E(X^r) = \beta\ {}_0\!\int^1 y^{-r/\alpha+1-1}\ (1-y)^{\beta+r/\alpha-1}\ dy$$

$$E(X^r) = \beta\ B(1-r/\alpha\ ,\ \beta+r/\alpha) \quad \text{for } \alpha > r$$

$$E[X^r] = \beta\ [\Gamma(1-r/\alpha)\ \Gamma(\beta+r/\alpha)]/\Gamma(\beta+1) \quad \text{for } \alpha>r \tag{6.5}$$

Therefore,

$$\text{Mean} = E(X) = [\Gamma(1-1/\alpha)\ \Gamma(\beta+1/\alpha)]/\Gamma(\beta) \quad \text{for } \alpha>1 \tag{6.6}$$

(ii) The graph of the probability density for different values of α is shown in figure (6.1).

(iii) The inverted Burr distribution is unimodel and the mode is given by the equation

$$X_{mode} = [(\alpha\beta-1)/(\alpha+\beta)]^{1/\alpha} \quad \alpha\beta > 1 \text{ and } \alpha,\beta > 0$$

(iv) Cumulative Frequency Function:

The cumulative frequency function gives the expected number of cases less than a given value. Hence expected frequencies in any given range are found simply by taking the difference between two values of this function. The cumulative frequency function or the distribution function G(x) of the inverted Burr distribution is

$$G(x) = {}_0\!\int^x g(y)\ dy$$

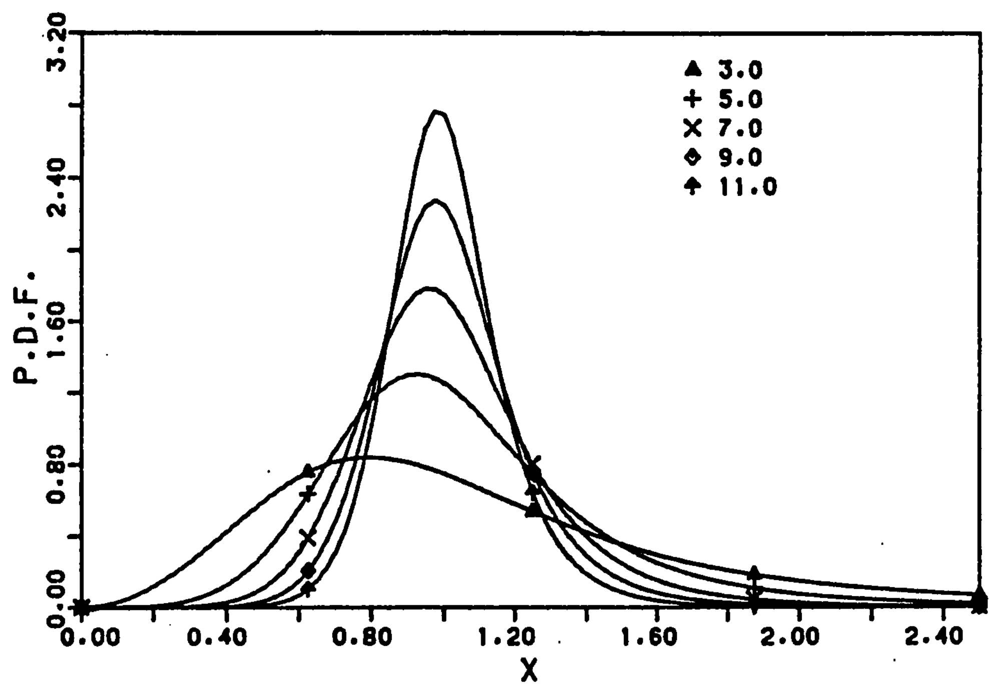

Fig. 6.1 The inverted Burr probability density function.

$$G(x) = \alpha\beta \int_0^x (1+y^\alpha)^{-1-\beta}\, dy \quad \text{where } x,\alpha,\beta > 0 \tag{6.7}$$

Using the transformation $z = (1 + y^\alpha)^{-1}$, for which the jacobian is

$$dy/dz = -(1/z - 1)^{1/\alpha - 1} / (\alpha z^2) \quad \text{where } 1 < z < (1 + x^\alpha)^{-1}$$

substituting values in (4.71) and after some manipulations we obtain

$$G(x) = \beta \int_z^1 (1-z)^{\beta-1}\, dz$$

where $z = (1+x^\alpha)^{-1}$

$$G(x) = [x^\alpha/(1+x^\alpha)]^\beta \quad \text{for } \alpha,\beta,x > 0 \tag{6.8}$$

(v) Median Of The Distribution:

Using (6.8), the median is the solution of the equation

$$X_{med} = (2^{1/\beta} - 1)^{1/\alpha}$$

The expression for the median shows that when $\beta = 1$ the inverted Burr and Burr distributions have their medians at 1 for all values of α.

(vi) Parametric Point Estimations

(a) Maximum Likelihood Esimates of The Parameters α and β:

Let $X_1,\ldots,X_n$ be a random sample from the inverted Burr population having probability density function

$$g(x) = \alpha\beta(1 + x^\alpha)^{-\beta-1} x^{\alpha\beta-1}, \quad x,\alpha,\beta > 0 \tag{6.9}$$

The likelihood function is

$$L(\alpha,\beta) = (\alpha\beta)^n \prod_{i=1}^{n} (x_i)^{\alpha\beta-1} / [\prod_{i=1}^{n} (1 + x_i^\alpha)^{1+\beta}]$$

Equating $\partial/\partial\alpha(\ln L)$ and $\partial/\partial\beta(\ln L)$ to zero we obtain the estimating equations for α and β as

$$n + \hat{\alpha}\hat{\beta}\sum_{i=1}^{n} \ln x_i - \hat{\beta}\sum_{i=1}^{n} \ln(1 + x_i) = 0 \tag{6.10}$$

$$\begin{aligned} n &+ \hat{\alpha}\hat{\beta}\sum_{i=1}^{n} \ln x_i - \hat{\alpha}^2\sum_{i=1}^{n} \ln x_i \,(x_i)^{\hat{\alpha}}/[(1 + x_i)] \\ &-\hat{\alpha}^2\hat{\beta}\sum_{i=1}^{n} (x_i)^{\hat{\alpha}} \ln x_i/[(1 + x_i^{\hat{\alpha}})] = 0 \end{aligned} \tag{6.11}$$

For a given sample, the maximum likelihood estimates of the parameters can be found using equations (6.10) and (6.11). These estimates are not in closed form, however, these can be solved numerically.

(b) Moments Estimates Of α and β:

The moment estimates of α and β are

$$(1/n)\sum_{i=1}^{n} x_i = [\Gamma(1-1/\tilde{\alpha}) \,.\, \Gamma(\tilde{\beta}+1/\tilde{\alpha})]/\Gamma(\tilde{\beta}) \quad \text{for } \tilde{\alpha} > 1 \text{ and}$$

$$(1/n)\sum_{i=1}^{n} (x_i)^2 = [\Gamma(1-2/\tilde{\alpha}) \,.\, \Gamma(\tilde{\beta}+2/\tilde{\alpha})]/\Gamma(\tilde{\beta}) \quad \text{for } \tilde{\alpha} > 2.$$

These equations can be solved numerically to find the moments estimates of α and β. Table (6.1) shows some estimates of α and β for different values of the statistics involved.

6.2 Moments Of The ith Order Statistic Of A Sample Drawn From The Inverted Burr Population

Suppose a random sample of size n is drawn from an inverted Burr population. Let $X_{(1)},\ldots,X_{(n)}$ be the order statistics of the sample drawn. i.e.

$$0 \le X_{(1)} \le X_{(2)} \le\ldots\le X_{(n)} < +\infty$$

The density function of the ith order statistic is given by

$$f(y) = n!/[(i-1)!(n-i)!]\ [G(y)]^{i-1}\ [1 - G(y)]^{n-i}\ g(y),$$

where $g(y)= \alpha\beta(1 + y^{\alpha})^{-1-\beta}\ y^{\alpha\beta-1}$ $\alpha,\beta>0$ and $0 \le y>\infty$

$$\text{and} \qquad G(y) = [y^{\alpha}/(1 + y^{\alpha}]^{\beta}$$

The rth moment of the ith order statistic , given a sample of size n is denoted by $E(Y^r|n)$ and

$$E(Y^r|n) = n!/[(i-1).(n-i).]\ {}_0\!\int^{\infty} y^r[y^{\alpha}/(1+y^{\alpha})]^{\beta(i-1)}$$

$$[1-(y^{\alpha}/(1+y^{\alpha})^{\beta}]^{n-1}\ \alpha\beta\ y^{\alpha\beta-1}\ (1+y^{\alpha})^{-1-\beta}\ dy.$$

Which after series expansion of one of the factor of the integrand and some manipulations becomes

$$E(Y^r|n) = n!\ \alpha\beta/[(i-1)!(n-i)!] \sum_{j=0}^{i-1} (-1)^i ({}_j{}^{i}{-1})\ {}_0\!\int^{\infty}$$

$$y^{\alpha\beta i+\alpha\beta j+r-1}.\quad (1 + y^{\alpha})^{-\beta i-\beta j-1}\ dy \qquad (6.12)$$

Let $u = (1+y^{\alpha})^{-1}$ so that $y = [(1 - u)/u]^{1/\alpha}$

and the jacobian of the transformation is

$$dy/du = -(1/u - 1)^{1/\alpha-1}\ 1/(\alpha u^2) \quad \text{where } 0\le u \le 1$$

TABLE 6.1

α	β	$(1/n)\sum_{i=1}^{n} x_i$	$(1/n)\sum_{i=1}^{n} x_i^2-[(1/n)\sum_{i=1}^{n} x_i]^2$
2.5	1.5	1.61	3.43
3.0	2.0	1.61	1.43
3.5	2.5	1.59	0.79
4.0	3.0	1.56	0.50
4.5	3.5	1.53	0.34
5.0	4.0	1.50	0.25
5.5	4.5	1.47	0.18
6.0	5.0	1.45	0.14
6.5	5.5	1.43	0.11
7.0	6.0	1.41	0.09
7.5	6.5	1.39	0.07
8.0	7.0	1.37	0.06
8.5	7.5	1.36	0.05
9.0	8.0	1.34	0.04
9.5	8.5	1.33	0.04
10.0	9.0	1.32	0.03
10.5	9.5	1.31	0.03
11.0	10.0	1.30	0.02
11.5	10.5	1.29	0.02
12.0	11.0	1.28	0.02

Making use of the above transformation we get

$$E(Y^r|n) = \beta\ n!/[(i-1)!(n-i)!] \sum_{j=0}^{i-1} (-1)^j \binom{i-1}{j} B(1-r/\alpha$$

$$;\ \beta i + \beta j + r/\alpha) \qquad (6.13)$$

where $B(m,n) = {}_0\int^1 u^{m-1} (1-u)^{n-1}\, du$ and $r < n$.

It is clear from (6.13) that the moments of the ith order statistic of a sample drawn from inverted Burr population is some linear combination of beta functions.

6.3 The Moments And The Sampling Distribution Of The Median:

Since the cumulative function of inverted Burr distribution is in a closed form so we can find the explicit probability density function of the median of a sample of an odd size drawn from the inverted Burr population. Let the sample size be $n= 2m+1$ for $m>0$. and let X_m denote the median of the sample. Then the distribution of X_m is given by

$$h(x_m)= [(2m+1)!/(m!)^2\ [G(x_m)]^m\ [1 - G(x_m)]^m\ g(x_m) \qquad (6.14)$$

where $G(x_m) = [x_m^{\alpha}/(1 + x_m^{\alpha})]^{\beta} \quad x_m > 0$

and $g(x_m) = \alpha\beta\ (1 + x_m^{\alpha})^{-1-\beta}\ x_m^{\alpha\beta-1}$.

Substituting for $G(x_m)$ and $g(x_m)$ in (6.14) we have

$$h(x_m) = [(2m+1)!/(m!)^2][1 - (x_m/(1+x_m^{\alpha}))^{\beta}]^m (x_m)^{\alpha\beta+\alpha\beta m-1}$$

$$(1 + x_m^{\alpha})^{-m\beta-\beta-1} \qquad x_m > 0 \qquad (6.15)$$

6.4 The Moments About The Origin Of The Median:

By definition $\mu'_r(X_m) = {}_0\int^{\infty} (x_m)^r\, h(x_m)\, dx_m$

$$= [(2m+1)!\ \alpha\beta/(m!)^2]\ {}_0\int^{\infty} (x_m)^{\alpha\beta+\alpha\beta m+r-1}/(1 + x_m^{\alpha})^{-m\beta-\beta-1}$$

$$[1-(x_m^{\alpha}/(1+x_m^{\alpha}))^{\beta}]^m\, dx_m$$

$$= [(2m+1)!\ \alpha\beta/(m!)^2] \sum_{j=0}^{m} (-1)^j \binom{m}{j}\ {}_0\int^{\infty}$$

$$\cdot\ (x_m)^{\alpha\beta j+\alpha\beta-1+r+\alpha\beta m}\ (1 + x_m^{\alpha})^{\beta j+1+\beta+m\beta}\, dx_m$$

Making transformation $y = (1 + x_m^{\alpha})^{-1}$ where $0<y<\infty$ and the jacobian of the transformation is

$$dx_m/dy = -(1/\alpha)(1 - y)^{1/\alpha-1}\ y^{-3}$$

Substituting values and after some simplifications we obtain

$$\mu'_r(X_m) = [(2m+1)!\ \beta/(m!)] \sum_{j=0}^{m} (-1)^j \binom{m}{j} B[2-(1+r)/\alpha\ ;$$

$$\beta j + \beta + r/\alpha + m\beta)$$

where B(.;.) denote the beta function. The central and noncentral moments for the sample median x_m can be computed to find the moment ratios.

7. THE HAZARD FUNCTIONS

One of the important fields in statistics is concerned with life testing problems. A common man, scientist and engineer is interested in the operating life length of devices. In this case the operating time X is a random variable. A major problem is to select a suitable statistical model for the random variable X. Most of the probability models in the *generalized Pearson system* are life models. In this chapter we discuss the notion of 'hazard function' for the inverted class of distributions. A comparison based on the hazard function is made between the *Pearson* and *Inverted Pearson* models.

Before we proceed to the discussion of the hazard functions of the *Inverted Pearson* system it would be worthwhile to recapitulate the basic knowledge of some basic ideas. We have the following definitions:

Hazard Function:

Let X be a random variable representing the life length of a component. Let X be distributed according to the probability density $g(x,\theta)$. Then the hazard function of X is defined as

$$\lambda(x) = g(x,\theta)/(1-G(x)) \tag{7.1}$$

Where $G(x)$ is the distribution function of X. In reliability theory hazard function is also named as the failure rate. Before we discuss the hazard functions of the inverted family of distributions let us clarify the physical meanings of the function $\lambda(x)$. We write the formula in equation (7.1) in the form

$\lambda(x)\Delta x = [g(x)\Delta x\ N]/[(1-G(x))N]$ (7.2)

Where N = number of exemplaires of the object under test.

$g(x)\Delta x$ = probability of failure of the object in the time interval $[x,x+\Delta x]$.

$g(x)\Delta x\ N$ = average number of objects that fail in the time interval $[x,x+\Delta x]$.

$[1-G(x)]\ N$ = Average number of objects that do not fail during the time x.

Thus, $\lambda(x)\Delta x$ represents the ratio of the number of objects which fail during the time interval $[x,x+\Delta x]$ to the number of objects that do not fail up to time x.

7.1 PROBABILITY OF SURVIVAL

The hazard function is clearly related to the conditional mathematical expectation of the remaining life time with the supposition that the component did not fail in the interval $(0,x_p)$.We give the following theorem which gives the conditional probability of survival of a component in terms of the hazard function.

Theorem:7.1.1

Let X be a random variable representing the lifetime of an object. Let X be distributed according to the density function g(x) with distribution function G(x). If $P(X_c>x|x_p)$ represents the probability that the object will continue to function for a time Δx after it has functioned without failure for a time x_p. Then

$$P(X_c>x|x_p)= \exp[-\int_{x_p}^{(x_p+x)} \lambda(x)\, dx\ ] \quad (7.3)$$

Proof:

We first show that an arbitrary distribution function G(X) can be written as

$$G(x)= 1 - \exp[-{}_0\int^{x} \lambda(x)]dx \qquad (7.4)$$

where $\lambda(x)$ is the hazard function of X. The hazard function $\lambda(x)$ in equation (7.1) can be written as

$$\lambda(x)= [d/dx(G(x))]/[1-G(x)]$$

or $\quad \lambda(x)dx= dG(x)/[1-G(x)] = d[-\ln(1-G(x))]$

Integrating from 0 to x we obtain

$${}_0\int^{x} \lambda(x)\, dx= -\ln(1 - G(x))$$

which implies that

$$G(x)= 1 - \exp[-{}_0\int^{x} \lambda(x)\, dx]$$

Now, using the definition of conditional probability

$$P(X_c>x|x_p)= P(X_c>x+x_p)/p(X_c>x_p)$$

which by the application of equation (7.4) can be written as

$$P(X_c>x|x_p)= [\exp(-{}_0\int^{(x+x_p)} \lambda(x)\, dx)]/[\exp(-{}_0\int^{x_p} \lambda(x)\, dx)]$$

$$= \exp[-{}_{x_p}\int^{(x+x_p)} \lambda(x)\, dx]$$

Which completes the proof.

Note that equation (7.3) can be used to establish relationship between reliability function and hazard function.

Corolary:7.1.2

Suppose that Δx denote the value of x when x is small, then

$$P(X_c > x | x_p) = \exp[-\lambda(x_p)\Delta x] \tag{7.5}$$

A distinguishing characteristic of a model is its particular hazard function $\lambda(x)$. The hazard functions of the *Pearson family* are discussed in detail in texts on statistical models e.g.[44]. Using the definition through equation (7.1), the hazard function of the *inverted weibull* distribution is

$$\lambda(x) = [abx^{-b-1}]/[\exp(ax^{-b} - 1)] \quad 0<x<\infty$$

Definition:

Let $L(x_p)$ be a random variable which represents the lifetime from some fixed instant x_p until the instant of failure under the condition that no failure occured prior to the instant x_p. $L(x_p)$ is called the residual life and the expected value of $L(x_p)$ is called the average residual life.

Using the L'hopital,s rule , we find the asymptotic value of the hazard function of the *inverted Weibull* distribution.

$$\lim_{x\to\infty} \lambda(x) = \lim_{x\to\infty} [abx^{-b-1}]/[\exp(ax^{-b} - 1)] = 0 \tag{7.6 }$$

where for weibull probability model

$$\lambda(x) = abx^{b-1} \tag{7.7}$$

and the asymptotic value of the hazard function depends upon the value of the shape parameter b. We have the following three cases

If $b= 1$ then $\lim_{x\to+\infty} \lambda(x) = a$ which is constant. (7.8)

If $b> 1$ then $\lim_{x\to+\infty} \lambda(x) = +\infty$ (7.9)

If $b< 1$ then $\lim_{x\to+\infty} \lambda(x) = 0$ (7.10)

7.2 COMPARISON OF VARIOUS HAZARD FUNCTIONS

The hazard function of Weibull model for different values of the shape parameter is depicted in Fig.(7.1) which show that the function $\lambda(x)$ decreases when x is increasing with shape paramẽter b<1. The function λ is constant for b=1, such a hazard function is called a *memoryless* hazard function, it shows that the deterioration of the object or component can not occur with operating time. i.e. the history of the component has no effect on the probability of failure, and when b>1 λ increases with the increasing values of x. The Weibull model is used in engineering sciences e.g. in the analysis of extreme value phenomena etc. and reliability engineering. On the other hand, the hazard function of *Inverted Weibull* distribution behaves (fig.7.2) very similar to the hazards of the Bernstein distribution(inverted normal distribution is a special case of Bernstien distribution) and log_normal distribution. The hazard function of the Bernstein distribution , logarithmic distribution and inverted Weibull distribution which correspond to a linear wear, which increase with the increase in x up to some instant x_0 and then start decreasing after the instant x_0. In reliability theory it corresponds to the fact that the items possesing high rate of wear can fail and the rate of wear of the surviving items is relatively small so that they have greater longevity.

The hazard function of the Bernstein distribution is

$$\lambda(x)=\frac{\exp[-(x-c)^2/(2ax^2+2b)]\quad(b+cax)}{(2\pi)^{1/2}[1-\Phi((x-c)/(ax^2+b))]\quad(b+ax^2)^{3/2}}$$

where Φ is N(0,1).

and $\lim_{x\to+\infty} \lambda(X)= 0$

Similarly, the hazard function of the logarithmical normal distribution is

$$\lambda(x)=\frac{A\ \exp[-(\log x-c)^2/(2\sigma^2)}{(2\pi)^2\sigma x[1-\Phi(\log x-c)/\sigma]}$$

We can easily see that $\lim_{x\to\infty} \lambda(x)= 0$

For *Inverted Weibull* , Bernstien and logarithmic normal distributions, the mean residual life time increases after the instant x_0. Like log_normal and Bernstien model; *inverted Weibull* is an other model of material strength and can prove equally good as other prominent models in the following areas

(1) Mortality Studies:(Human life behavour)

(2) Medicine Sciences And Diseases Control to study the effect of a certain disease, where after some time x_0 a part of population develop immunity to certain disease, and failure rate $\lambda(x)$ start decreasing.

(3) Life Testing which is composed of wear out failure till time x_0, after x_0 a strengthening phenomena of material is observed such as crack arrest in the fatigue behavour of ductile material.

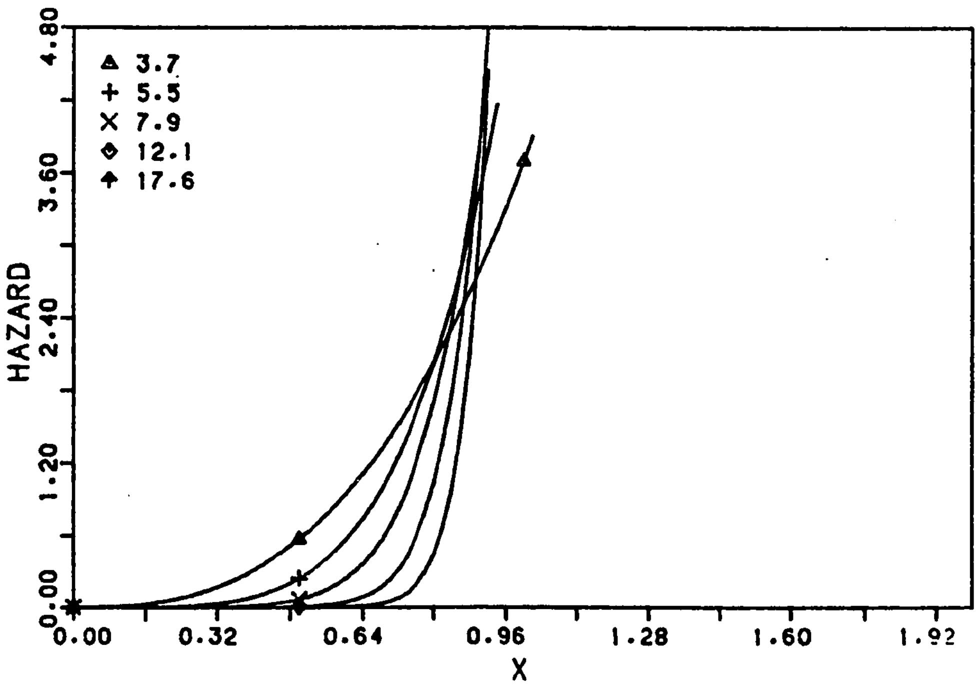

Fig. 7.1 The Weibull hazard function.

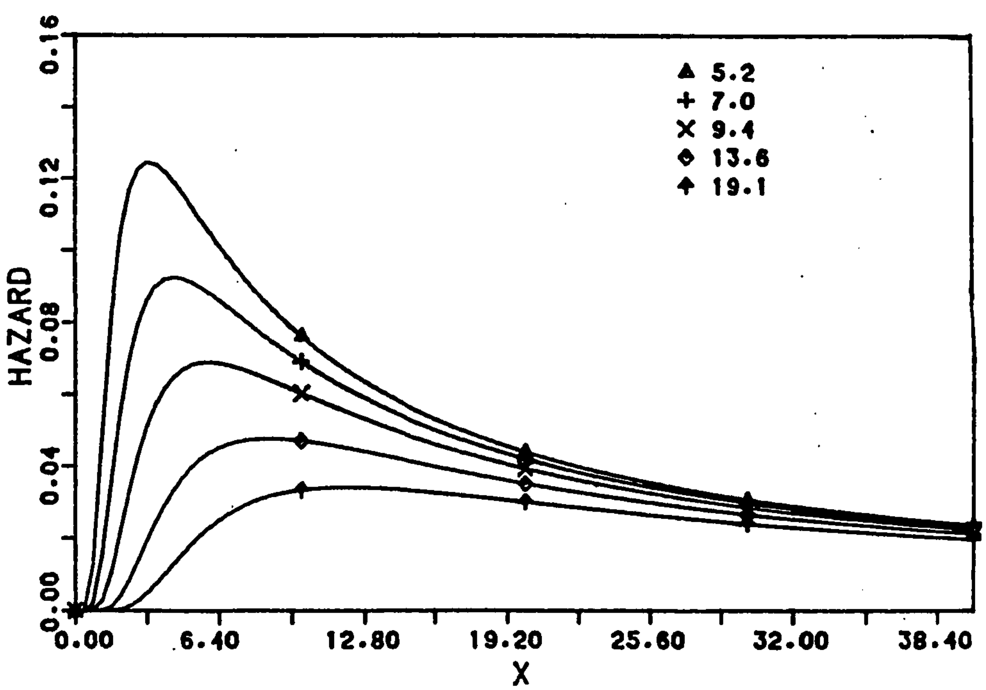

Fig. 7.2 The inverted Weibull hazard function.

The value of x_0 is the value of x given by the solution of the equation

$$x^b(1 - \mathrm{Exp}(-ax^{-b}) = ab/(b+1)$$

From the preceeding discussion we notice that the hazard functions of *inverted Weibull* and Weibull distributions are significantly different in shape and characterstics. It is interesting to note that for shape parameter $\lambda>4$, the inverse Gaussian and *inverted inverse Gaussian* models provide almost similar hazard functions, this is due to similarities of $g_X(x)$ and $g_{1/X}(x)$. Figures (7.3) and (7.4) show the hazard function of *inverted inverse Gaussian and inverse Gaussian* distributions for different values of the shape parameter λ while the scale parameter μ is fixed and equals to one.

The relationship between the shape parameter b and the point x_0 of maximum hazard is shown for *inverted Weibull* model in Fig.(7.5). The hazard function of inverted Burr distribution is

$$\lambda(x)= \alpha\beta\ x^{\alpha\beta-1}/[(1+x^{\alpha})^{\beta+1}-x^{\alpha\beta}(1+x^{\alpha})]$$

Figure (7.6) shows the graph of $\lambda(x)$ for different values of β where $\alpha=1.0$.

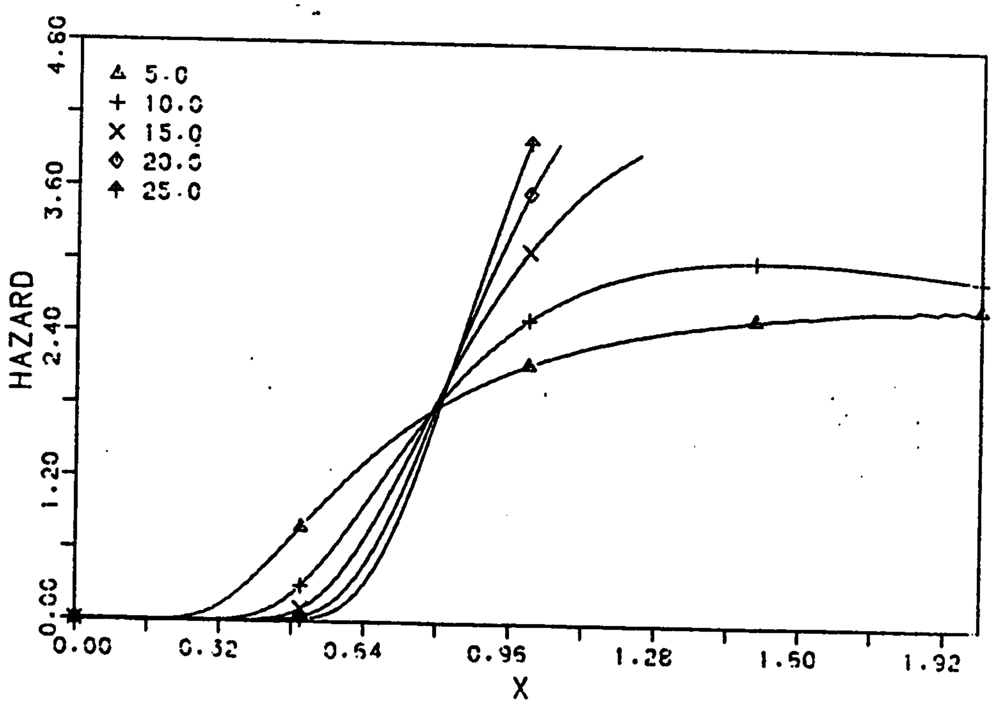

Fig. 7.4 The inverse Gaussian hazard function.

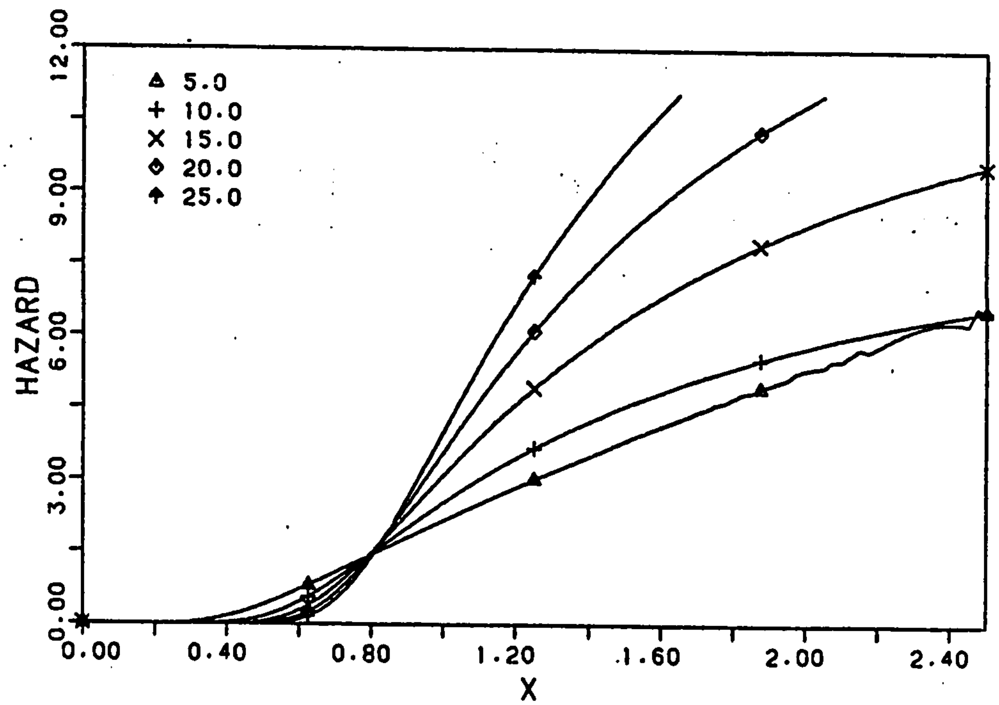

Fig. 7.3 The inverted inverse Gaussian hazard function.

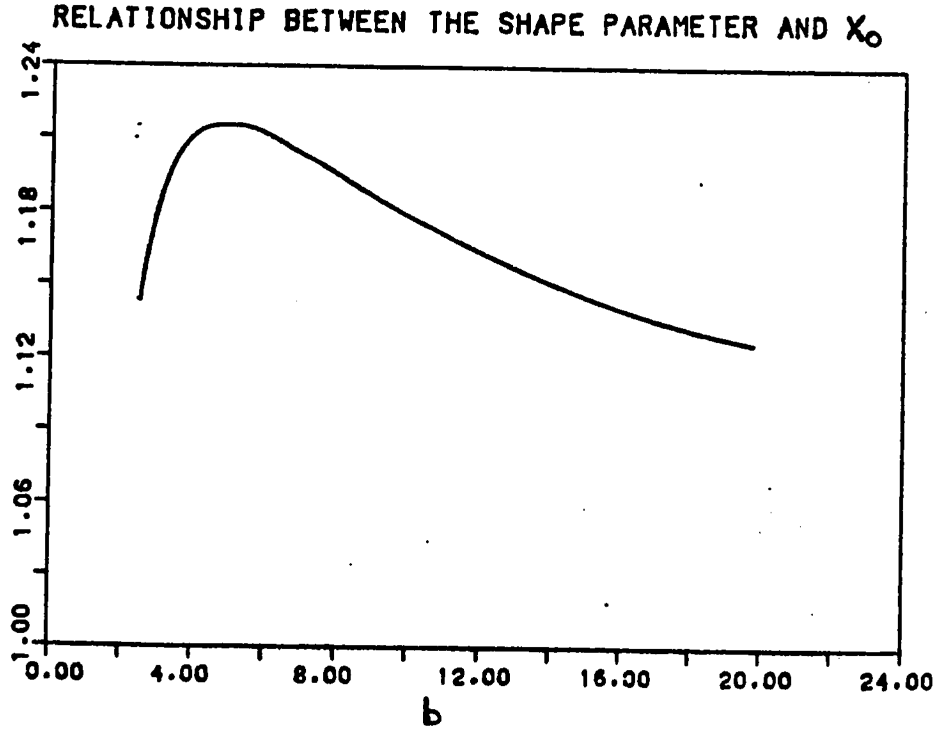

Fig. 7.5 Relationship between the shape parameter b and the point x_0.

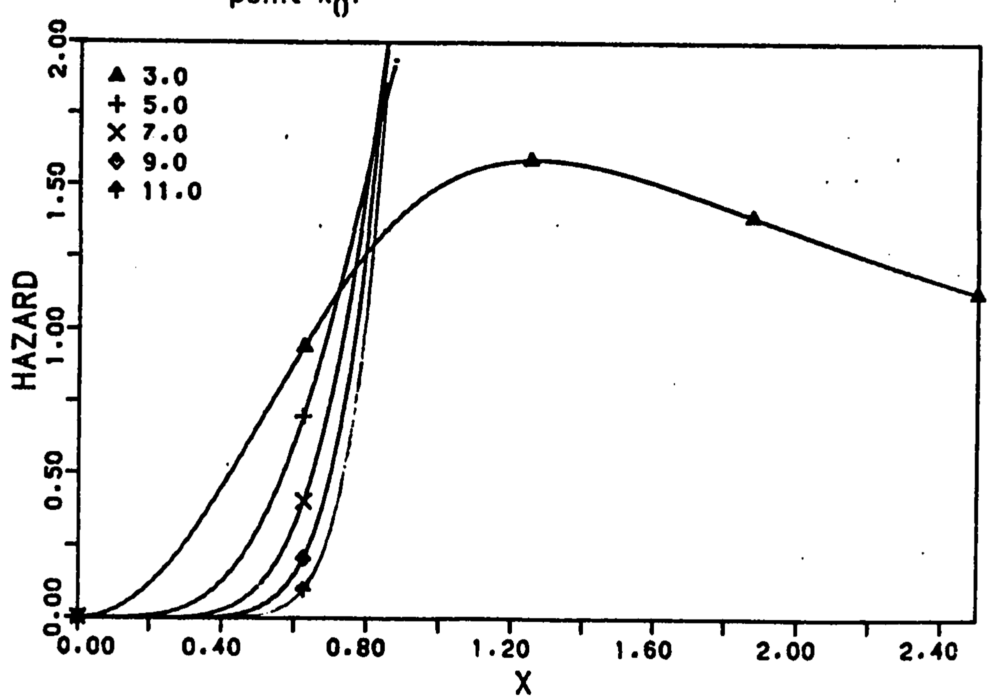

Fig. 7.6 Inverted Burr hazard function.

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